# Package ‘Directional’

May 7, 2026

**Type** Package

**Title** A Collection of Functions for Directional Data Analysis

**Version** 7.5

**Date** 2026-04-29

**Maintainer** Michail Tsagris <mtsagris@uoc.gr>

**Description** A collection of functions for directional data (including massive data, with millions of observations) analysis.
Hypothesis testing, discriminant and regression analysis, MLE of distributions and more are included.

The standard textbook for such data is the ``Directional Statistics'' by Mardia, K. V. and Jupp, P. E. (2000).

Other references include:

a) Paine J.P., Preston S.P., Tsagris M. and Wood A.T.A. (2018). ``An elliptically symmetric angular Gaussian distribution''. Statistics and Computing 28(3): 689-697. <doi:10.1007/s11222-017-9756-4>.

b) Tsagris M. and Alenazi A. (2019). ``Comparison of discriminant analysis methods on the sphere''. Communications in Statistics: Case Studies, Data Analysis and Applications 5(4):467--491. <doi:10.1080/23737484.2019.1684854>.

c) Paine J.P., Preston S.P., Tsagris M. and Wood A.T.A. (2020). ``Spherical regression models with general covariates and anisotropic errors''. Statistics and Computing 30(1): 153--165. <doi:10.1007/s11222-019-09872-2>.

d) Tsagris M. and Alenazi A. (2024). ``An investigation of hypothesis testing procedures for circular and spherical mean vectors''. Communications in Statistics-Simulation and Computation, 53(3): 1387--1408. <doi:10.1080/03610918.2022.2045499>.

e) Yu Z. and Huang X. (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statistics, 18(1): 301--334. <doi:10.1214/23-EJS2210>.

f) Tsagris M. and Alzeley O. (2025). ``Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling''. Australian & New Zealand Journal of Statistics, 67(1): 77--103. <doi:10.1111/anzs.12434>.

g) Tsagris M., Papastamoulis P. and Kato S. (2025). ``Directional data analysis: spherical Cauchy or Poisson kernel-based distribution''. Statistics and Computing, 35:51. <doi:10.1007/s11222-025-10583-0>.

h) Alzeley O. and Tsagris (2026). On the generalized circular projected Cauchy distribution. <doi:10.48550/arXiv.2603.04030>.

**License** GPL (>= 2)

**Imports** bigstatsr, doParallel, foreach, ggplot2, grDevices, magrittr, parallel, Rfast, Rfast2, Rnanoflann, rgl, rnaturalearth, sf

**Suggests** bigreadr

**RoxygenNote** 6.1.1

**NeedsCompilation** no

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# Contents

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`Directional-package` *This is an R package that provides methods for the statistical analysis of directional data, including massive (very large scale) directional data.*

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## Description

Circular-linear regression, spherical-spherical regression, spherical regression, discriminant analysis, ANOVA for circular and (hyper-)spherical data, tests for eaquality of conentration parameters, maximum likelihood estimation of the parameters of many distributions, random values generation from various distributions, contour plots and many more functions are included.

## Details

| | |
|---|---|
| Package: | Directional |
| Type: | Package |
| Version: | 7.5 |
| Date: | 2026-04-29 |
| License: | GPL-2 |

## Maintainers

Michail Tsagris <mtsagris@uoc.gr>.

## Note

**Acknowledgments:**

Professor Andy Wood and Dr Simon Preston from the university of Nottingham are highly appreciated for being my supervisors during my post-doc in directional data analysis.

Dr Georgios Pappas (former postDoc at the university of Nottingham) helped me construct the contour plots of the von Mises-Fisher and the Kent distribution.

Dr Christopher Fallaize and Dr Theo Kypraios from the university of Nottingham have provided a function for simulating from the Bingham distribution using rejection sampling. So any questions regarding this function should be addressed to them.

Dr Kwang-Rae Kim (post-doc at the university of Nottingham) answered some of my questions.

Giorgos Borboudakis (PhD student at the university of Crete) pointed out to me a not so clear message in the algorithm of generating random values from the von Mises-Fisher distribution.

Panagiotis (pronounced Panayiotis) Tzirakis (master student at the department of computer science in Heraklion during the 2013-2015 seasons) showed me how to perform parallel computing in R and he is greatly acknowledged and appreciated not only from me but from all the readers of this document. He also helped me with the vectorization of some contour plot functions.

Professor John Kent from the university of Leeds is acknowledged for clarifying one thing with the ovalness parameter in his distribution.

Phillip Paine (postdoc at the university of Nottingham) spotted that the function `rfb` is rather slow and he suggested me to change it. The function has changed now and this is also due to Joshua Davis (from Carleton College, Northfield, MN) who spotted that mistakes could occur, due a vector not being a matrix.

Professor Kurt Hornik from the Vienna university of economics and business is greatly acknowledged for his patience and contast help with this (and not only) R package.

Manos Papadakis is also acknowledged for his programming tips and for his assistance with the "htest" class object.

Dr Mojgan Golzy spotted a mistake in the function `desag` and Michail is very happy for that.

Lisette de Jonge-Hoekstra from the University of Groningen found a wrong sentence in the help file of function `spml.reg` which is now deleted.

Peter Harremoes from the Copenhagen Business College spotted a mistake in the confidence interval of the function `circ.summary` which has now been corrected.

Dr Gregory Emvalomatis from the University of Crete helped me understand better the EM algorithm for mixture models and I fixed a bug in the function `mixvmf.mle`.

Kinley Russell, PhD student at the Johns Hopkins University School of Medicine, suggested that I include bootstrap ANOVA functions.

Sia Ahmadi found a mistake in the function `conc.test` which has now been corrected.

Rafail Vargiakakis found a bug in the functions `circbeta.mle`, `circexp.mle` and `mmvm.mle`.

Arhtur Pewsey spotted a bug in the function `cardio.mle` which is now fixed.

Nicholas Foster spotted no output in the functions `kuiper` and `watson`. The same problem occured in two more functions.

If you want more information on many of these algorithms see Chapters 9 and 10 in the following document. https://www.researchgate.net/publication/324363311_Multivariate_data_analysis_in_R

### Author(s)

Michail Tsagris <mtsagris@uoc.gr>, Giorgos Athineou <gioathineou@gmail.com>, Christos Adam <pada4m4@gmail.com>, Zehao Yu <zehaoy@email.sc.edu>, Anamul Sajib <sajibstat@du.ac.bd>, Eli Amson <eli.amson1988@gmail.com>, Micah J. Waldstein <micah@waldste.in> and Panagiotis Papastamoulis <papastamoulis@aueb.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley and Sons.

`(Hyper-)spherical regression using rotational symmetric distributions`
*(Hyper-)spherical regression using the rotational symmetric distributions*

## Description

(Hyper-)spherical regression using the rotational symmetric distributions.

## Usage

```
vmfreg(y, x, con = TRUE, xnew = NULL, tol = 1e-06)
spcauchy.reg(y, x, con = TRUE, xnew = NULL, tol = 1e-06)
pkbd.reg(y, x, con = TRUE, xnew = NULL, tol = 1e-6)
pkbd.reg2(y, x, con = TRUE, xnew = NULL, tol = 1e-6)
```

## Arguments

| | |
|---|---|
| `y` | A matrix with any number of columns containing the (unit vector) (hyper-)spherical data. |
| `x` | The predictor variable(s), they can be continnuous, (hyper-)spherical, categorical or a mix of them. |
| `con` | Do you want the constant term in the regression? |
| `xnew` | If you have new data use it, otherwise leave it NULL. |
| `tol` | A tolerance value to decide when to stop the successive optimaizations. |

## Details

The second parametrization of the projected normal and of the von Mises-Fisher regression (Paine et al., 2020) is applied. The same is true for the SIPC distribution. For more information see the paper by Paine et al. (2020). The difference from `vmf.reg` is that the latter is designed for the sphere only, whereas this function works in the hyper-sphere also.

As for the spcauchy.reg() and pkbd.reg() they are based upon the spherical Cauchy (Kato and McCullagh, 2020) and the Poisson kernel-based (Golzy and Markatou, 2020) distributions. These two use Newton-Raphson, but the pkbd.reg2() uses the `optim`. We have noticed some numerical issues with the pkbd.reg() when the dimensionalities of the variables are large and this is why we also provide the (much slower) pkbd.reg2() function.

## Value

A list including:

| | |
|---|---|
| `runtime` | The runtime of the regression. |
| `iters` | The number of iterations required until convergence of the Newton-Raphson algorithm. |

| | |
|---|---|
| loglik | The log-likelihood of the regression model. |
| fit | This is a measure of fit of the estimated values, defined as $\sum_{i=1}^n y_i^T \hat{y}_i$. This appears if the argument "xnew" is NULL. |
| beta | The beta coefficients. |
| seb | The standard error of the beta coefficients. |
| ki | The norm of the fitted values. In the von Mises-Fisher regression this is the concentration parameter of each observation. This is returned if the argument "xnew" is NULL. |
| g2 | The norm of the fitted values. In the spherical Cauchy and the PKBD regression this is the concentration parameter of each observation. This is returned if the argument "xnew" is NULL. |
| est | The fitted values of xnew if "xnew" is NULL. If it is not NULL, the fitted values for the "xnew" you supplied will be returned. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

P. J. Paine, S. P. Preston, M. Tsagris and Andrew T. A. Wood (2020). Spherical regression models with general covariates and anisotropic errors. Statistics and Computing, 30(1): 153–165. https://link.springer.com/content/pdf/10.1007

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248.

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

## See Also

esag.reg, vmf.reg, spml.reg

## Examples

```
y <- rvmf(150, rnorm(5), 5)
a <- vmfreg(y, iris[, 1])
b <- spcauchy.reg(y, iris)
```

## A test for testing the equality of the concentration parameters for ciruclar data

*A test for testing the equality of the concentration parameter among g samples, where g >= 2 for ciruclar data*

### Description

A test for testing the equality of the concentration parameter among g samples, where g >= 2 for ciruclar data. It is a tangential approach.

### Usage

```
tang.conc(u, ina, rads = FALSE)
```

### Arguments

| | |
|---|---|
| `u` | A numeric vector containing the values of all samples. |
| `ina` | A numerical variable or factor indicating the groups of each value. |
| `rads` | If the data are in radians this should be TRUE and FALSE otherwise. |

### Details

This test works for circular data.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons. Fisher, N. I. (1995). Statistical analysis of circular data. Cambridge University Press.

## See Also

embed.circaov, hcf.circaov, lr.circaov, het.circaov, conc.test

## Examples

```
x <- rvonmises(100, 2.4, 15)
ina <- rep(1:4,each = 25)
tang.conc(x, ina, rads = TRUE)
```

---

# Angular central Gaussian random values simulation
*Angular central Gaussian random values simulation*

---

## Description

Angular central Gaussian random values simulation.

## Usage

```
racg(n, sigma)
```

## Arguments

| | |
|---|---|
| n | The sample size, a numerical value. |
| sigma | The covariance matrix in $R^d$. |

## Details

The algorithm uses univariate normal random values and transforms them to multivariate via a spectral decomposition. The vectors are then scaled to have unit length.

## Value

A matrix with the simulated data.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tyler D. E. (1987). Statistical analysis for the angular central Gaussian distribution on the sphere. Biometrika 74(3): 579–589.

## See Also

acg.mle, rvmf, rvonmises

### Examples

```
s <- cov( iris[, 1:4] )
x <- racg(100, s)
Directional::acg.mle(x)
Directional::vmf.mle(x)
## the concentration parameter, kappa, is very low, close to zero, as expected.
```

---

## Anova for (hyper-)spherical data

*Analysis of variance for (hyper-)spherical data*

---

### Description

Analysis of variance for (hyper-)spherical data.

### Usage

```
hcf.aov(x, ina, fc = TRUE)
hclr.aov(x, ina)
lr.aov(x, ina)
embed.aov(x, ina)
het.aov(x, ina)
```

### Arguments

| | |
|---|---|
| `x` | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| `ina` | A numerical variable or a factor indicating the group of each vector. |
| `fc` | A boolean that indicates whether a corrected F test should be used or not. |

### Details

The high concentration (hcf.aov), high concentration log-likelihood ratio (hclr.aov), log-likelihood ratio (lr.aov), embedding approach (embed.aov) or the non equal concentration parameters approach (het.aov) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degree(s) of freedom of the test. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

hcf.boot, hcfboot, hclr.circaov,

### Examples

```
x <- rvmf(60, rnorm(3), 15)
ina <- rep(1:3, each = 20)
hcf.aov(x, ina)
hcf.aov(x, ina, fc = FALSE)
lr.aov(x, ina)
embed.aov(x, ina)
het.aov(x, ina)
```

---

## Anova for circular data

*Analysis of variance for circular data*

---

### Description

Analysis of variance for circular data.

### Usage

```
hcf.circaov(u, ina, rads = FALSE)
hclr.circaov(u, ina, rads = FALSE)
lr.circaov(u, ina, rads = FALSE)
het.circaov(u, ina, rads = FALSE)
embed.circaov(u, ina, rads = FALSE)
```

### Arguments

| | |
|---|---|
| `u` | A numeric vector containing the data. |
| `ina` | A numerical or factor variable indicating the group of each value. |
| `rads` | If the data are in radians, this should be TRUE and FALSE otherwise. |

### Details

The high concentration (hcf.circaov), high concentration likelihood ratio (hclr.aov), log-likelihood ratio (lr.circaov), embedding approach (embed.circaov) or the non equal concentration parameters approach (het.circaov) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degree(s) of freedom of the test. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

`hclr.aov`, `hcfcirc.boot`, `hcfcircboot`

## Examples

```
x <- rvonmises(100, 2.4, 15)
ina <- rep(1:4,each = 25)
hcf.circaov(x, ina, rads = TRUE)
lr.circaov(x, ina, rads = TRUE)
het.circaov(x, ina, rads = TRUE)
embed.circaov(x, ina, rads = TRUE)
hclr.circaov(x, ina, rads = TRUE)
```

---

# BIC for the model based clustering using mixtures of rotationally symmetric distributions

*BIC to choose the number of components in a model based clustering using mixtures of rotationally symmetric distributions*

---

## Description

BIC to choose the number of components in a model based clustering using mixtures of rotationally symmetric distributions

## Usage

```
bic.mixvmf(x, G = 5, n.start = , tol = 1e-6, maxiters = 500)
bic.mixspcauchy(x, G = 5, n.start = 5, tol = 1e-6, maxiters = 500)
bic.mixpkbd(x, G = 5, n.start = 5, tol = 1e-6, maxiters = 500)
```

## Arguments

| | |
|---|---|
| `x` | A matrix containing directional data. |
| `G` | The maximum number of clusters to be tested. Default value is 5. |
| `n.start` | The number of random starts to try. See also R's built-in function `kmeans` for more information about this. |
| `tol` | The tolerance value to terminate the EM algorithm. |
| `maxiters` | The maximum number of iterations to perform. |

## Details

The function computes the BIC (and ICL) to decide on the optimal number of clusters when using mixtures of von Mises-Fisher, mixtures of spherical Cauchy or mixtures of Poisson kernel-based distributions.

### Value

A plot of the BIC values and a list including:

| | |
|---|---|
| `bic` | The BIC values for all the models tested. |
| `icl` | The ICL values for all the models tested. |
| `runtime` | The run time of the algorithm. A numeric vector. The first element is the user time, the second element is the system time and the third element is the elapsed time. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris `<mtsagris@uoc.gr>`.

### References

Hornik, K. and Grun, B. (2014). movMF: An R package for fitting mixtures of von Mises-Fisher distributions. Journal of Statistical Software, 58(10): 1–31.

Biernacki C., Celeux G. and Govaert, G. (2000). Assessing a mixture model for clustering with the integrated completed likelihood. IEEE Transactions on Pattern Analysis and Machine Intelligence, 22(7): 719–725.

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

### See Also

`mixvmf.mle`, `rmixvmf`, `mixvmf.contour`

### Examples

```
x <- as.matrix( iris[, 1:4] )
x <- x / sqrt( rowSums(x^2) )
bic.mixvmf(x)
```

---

## `Bootstrap 2-sample mean test for (hyper-)spherical data`
*Bootstrap 2-sample mean test for (hyper-)spherical data*

---

### Description

Bootstrap 2-sample mean test for (hyper-)spherical data.

## Usage

```
hcf.boot(x1, x2, fc = TRUE, B = 999)
lr.boot(x1, x2, B = 999)
hclr.boot(x1, x2, B = 999)
embed.boot(x1, x2, B = 999)
het.boot(x1, x2, B = 999)
```

## Arguments

| | |
|---|---|
| `x1` | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| `x2` | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| `fc` | A boolean that indicates whether a corrected F test should be used or not. |
| `B` | The number of bootstraps to perform. |

## Details

The high concentration (hcf.boot), log-likelihood ratio (lr.boot), high concentration log-likelihood ratio (hclr.boot), embedding approach (embed.boot) or the non equal concentration parameters approach (het.boot) is used.

## Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are bootstrap based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

hcf.aov, hcf.perm, hcfboot

### Examples

```
x <- rvmf(60, rnorm(3), 15)
ina <- rep(1:2, each = 30)
x1 <- x[ina == 1, ]
x2 <- x[ina == 2, ]
hcf.boot(x1, x2)
lr.boot(x1, x2)
het.boot(x1, x2)
```

---

Bootstrap 2-sample mean test for circular data

## *Bootstrap 2-sample mean test for circular data*

---

### Description

Bootstrap 2-sample mean test for circular data.

### Usage

```
hcfcirc.boot(u1, u2, rads = TRUE, B = 999)
lrcirc.boot(u1, u2, rads = TRUE, B = 999)
hclrcirc.boot(u1, u2, rads = TRUE, B = 999)
embedcirc.boot(u1, u2, rads = TRUE, B = 999)
hetcirc.boot(u1, u2, rads = TRUE, B = 999)
```

### Arguments

| | |
|---|---|
| u1 | A numeric vector containing the data of the first sample. |
| u2 | A numeric vector containing the data of the first sample. |
| rads | If the data are in radians, this should be TRUE and FALSE otherwise. |
| B | The number of bootstraps to perform. |

### Details

The high concentration (hcfcirc.boot), the log-likelihood ratio test (lrcirc.boot), high concentration log-likelihood ratio (hclrcirc.boot), embedding approach (embedcirc.boot), or the non equal concentration parameters approach (hetcirc.boot) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are bootstrap based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

`hcf.circaov`, `hcfcircboot`, `het.aov`

### Examples

```
u1 <- rvonmises(20, 2.4, 5)
u2 <- rvonmises(20, 2.4, 10)
hcfcirc.boot(u1, u2)
```

---

`Bootstrap ANOVA for (hyper-)spherical data`

## *Bootstrap ANOVA for (hyper-)spherical data*

---

### Description

Bootstrap ANOVA for (hyper-)spherical data.

### Usage

```
hcfboot(x, ina, B = 999)
hetboot(x, ina, B = 999)
```

### Arguments

| | |
|---|---|
| `x` | A matrix with the combined data (from all groups) in Euclidean coordinates, i.e. unit vectors. |
| `ina` | The grouping variables. A factor or a numerical vector specifying the groups to which each observation belongs to. |
| `B` | The number of bootstraps to perform. |

### Details

The high concentration (hcfboot), or the non equal concentration parameters approach (hetboot) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are bootstrap based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

`hcf.boot`, `hcf.aov`

### Examples

```
x <- rvmf(60, rnorm(3), 10)
ina <- rep(1:3, each = 20)
hcfboot(x, ina)
```

---

## Bootstrap ANOVA for circular data
*Bootstrap ANOVA for circular data*

---

### Description

Bootstrap ANOVA for circular data.

### Usage

```
hcfcircboot(u, ina, rads = TRUE, B = 999)
hetcircboot(u, ina, rads = TRUE, B = 999)
```

### Arguments

| | |
|---|---|
| `u` | A numeric vector containing the data of all groups. |
| `ina` | The grouping variables. A factor or a numerical vector specifying the groups to which each observation belongs to. |
| `rads` | If the data are in radians, this should be TRUE and FALSE otherwise. |
| `B` | The number of bootstraps to perform. |

### Details

The high concentration (hcfcircboot), or the non equal concentration parameters approach (hetcircboot) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are bootstrap based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

hcf.circaov, het.aov

### Examples

```
u1 <- rvonmises(20, 2.4, 5)
u2 <- rvonmises(20, 2.4, 10)
hcfcirc.boot(u1, u2)
```

---

## Check visually whether matrix Fisher samples is correctly generated or not

*Check visually whether matrix Fisher samples is correctly generated or not.*

---

### Description

It plots the log probability trace of matrix Fisher distribution which should close to the maximum value of the logarithm of matrix Fisher distribution, if samples are correctly generated.

### Usage

```
visual.check(x, Fa)
```

### Arguments

| | |
|---|---|
| x | The simulated data. An array with at least 2 3x3 matrices. |
| Fa | An arbitrary 3x3 matrix represents the parameter matrix of this distribution. |

### Details

For a given parameter matrix Fa, maximum value of the logarithm of matrix Fisher distribution is calculated via the form of singular value decomposition of $Fa = U\Lambda V^T$ which is $tr(\Lambda)$. Multiply the last column of $U$ by $-1$ and replace small eigenvalue, say, $\lambda_3$ by $-\lambda_3$ if $|UV^T| = -1$.

### Value

A plot which shows log probability trace of matrix Fisher distribution. The values are also returned.

### Author(s)

Anamul Sajib.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

### References

Habeck M. (2009). Generation of three-dimensional random rotations in fitting and matching problems. Computational Statistics, 24(4):719–731.

### Examples

```
Fa <- matrix( c(85, 11, 41, 78, 39, 60, 43, 64, 48), ncol = 3) / 10
x <- rmatrixfisher(1000, Fa)
a <- visual.check(x, Fa)
```

---

## `Circular correlations between one and many circular variables` *Circular correlations between two circular variables*

---

### Description

Circular correlations between two circular variables.

### Usage

```
circ.cors1(theta, phi, rads = FALSE)
circ.cors2(theta, phi, rads = FALSE)
```

### Arguments

| | |
|---|---|
| `theta` | The first cirular variable expressed in radians, not degrees. |
| `phi` | The other cirular variable. In the case of "circ.cors1" this is a matrix with many circular variables. In either case, the values must be in radians, not degrees. |
| `rads` | If the data are expressed in rads, then this should be TRUE. If the data are in degrees, then this is FALSE. |

### Details

Correlation for circular variables using the cosinus and sinus formula of Jammaladaka and Sen-Gupta (1988).

### Value

A matrix with two columns, the correlations and the p-values.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Jammalamadaka, R. S. and Sengupta, A. (2001). Topics in circular statistics. World Scientific.

Jammalamadaka, S. R. and Sarma, Y. R. (1988). A correlation coefficient for angular variables. Statistical Theory and Data Analysis, 2:349–364.

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

spml.reg

### Examples

```
y <- runif(50, 0, 2 * pi)
x <- matrix(runif(50 * 10, 0, 2 * pi), ncol = 10)
circ.cors1(y, x, rads = TRUE)
```

---

`Circular correlations between two circular variables`

## *Circular correlations between two circular variables*

---

### Description

Circular correlations between two circular variables.

### Usage

```
circ.cor1(theta, phi, rads = FALSE)

circ.cor2(theta, phi, rads = FALSE)
```

### Arguments

| | |
|---|---|
| theta | The first cirular variable. |
| phi | The other cirular variable. |
| rads | If the data are expressed in rads, then this should be TRUE. If the data are in degrees, then this is FALSE. |

### Details

circ.cor1: Correlation for circular variables using the cosinus and sinus formula of Jammaladaka and SenGupta (1988).

circ.cor2: Correlation for circular variables using the cosinus and sinus formula of Mardia and Jupp (2000).

### Value

A vector including:

| | |
|---|---|
| rho | The value of the correlation coefficient. |
| p-value | The p-value of the zero correlation hypothesis testing. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Jammalamadaka, R. S. and Sengupta, A. (2001). Topics in circular statistics. World Scientific.

Jammalamadaka, S. R. and Sarma, Y. R. (1988) . A correlation coefficient for angular variables. Statistical Theory and Data Analysis, 2:349–364.

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

circlin.cor, circ.cor2, spml.reg

### Examples

```
y <- runif(50, 0, 2 * pi)
x <- runif(50, 0, 2 * pi)
circ.cor1(x, y, rads = TRUE)
circ.cor2(x, y, rads = TRUE)
```

`Circular distance correlation between two circular variables`
*Circular distance correlation between two circular variables*

## Description

Circular distance correlation between two circular variables.

## Usage

```
circ.dcor(theta, phi, rads = FALSE)
```

## Arguments

| | |
|---|---|
| `theta` | The first cirular variable. |
| `phi` | The other cirular variable. |
| `rads` | If the data are expressed in rads, then this should be TRUE. If the data are in degrees, then this is FALSE. |

## Details

The angular data are transformed to their Euclidean coordinates and then the distance correlation is computed.

## Value

A list including:

| | |
|---|---|
| `dcov` | The distance covariance. |
| `dvarX` | The distance variance of x. |
| `dvarY` | The distance variance of Y. |
| `dcor` | The distance correlation. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

## References

G.J. Szekely, M.L. Rizzo and N. K. Bakirov (2007). Measuring and Testing Independence by Correlation of Distances. Annals of Statistics, 35(6):2769-2794.

## See Also

`circlin.cor`, `circ.cor2`, `spher.dcor`

### Examples

```
y <- runif(50, 0, 2 * pi)
x <- runif(50, 0, 2 * pi)
circ.dcor(x, y, rads = TRUE)
```

---

## Circular or angular regression

*Circular or angular regression*

---

### Description

Regression with circular dependent variable and Euclidean or categorical independent variables.

### Usage

```
spml.reg(y, x, rads = TRUE, xnew = NULL, seb = FALSE, tol = 1e-07)
circpurka.reg(y, x, rads = TRUE, xnew = NULL)
cipc.reg(y, x, rads = TRUE, xnew = NULL, tol = 1e-06)
gcpc.reg(y, x, rads = TRUE, xnew = NULL)
```

### Arguments

| | |
|---|---|
| y | The dependent variable, a numerical vector, it can be in radians or degrees. |
| x | The independent variable(s). Can be Euclidean or categorical (factor variables). |
| rads | If the dependent variable is expressed in rads, this should be TRUE and FALSE otherwise. |
| xnew | The new values of some independent variable(s) whose circular values you want to predict. Can be Euclidean or categorical. If they are categorical, the user must provide them as dummy variables. It does not accept factor variables. If you have no new x values, leave it NULL (default). |
| seb | a boolean variable. If TRUE, the standard error of the coefficients will be be returned. Set to FALSE in case of simulation studies or in other cases such as a forward regression setting for example. In these cases, it can save some time. |
| tol | The tolerance value to terminate the Newton-Raphson algorithm. |

### Details

For the spml.reg(), the Newton-Raphson algorithm is fitted in this regression as described in Presnell et al. (1998). For the cipc.reg(), the Newton-Raphson algorithm is fitted in this regression as described in Tsagris and Alenazy (2023). Note that the cipc.reg() is the same as the wrapped Cauchy regression. For the circpurka.reg() the optim() function is employed. For the gcpc.reg() the optim() and the optimise() functions are being used.

## Value

A list including:

- `runtime` The runtime of the procedure.
- `iters` The number of iterations required until convergence of the Newton-Raphson algorithm.
- `beta` The regression coefficients.
- `seb` The standard errors of the coefficients.
- `loglik` The value of the maximised log-likelihood.
- `est` The fitted values expressed in radians if the obsereved data are in radians and in degrees otherwise. If xnew is not NULL, i.e. if you have new x values, then the predicted values of y will be returned.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

## References

Presnell B., Morrison S. P. and Littell Ramon C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Alzeley O. & Tsagris M. (2026). On the generalized circular projected Cauchy distribution. https://arxiv.org/pdf/2603.04030.

## See Also

`circlin.cor`, `circ.cor1`, `circ.cor2`, `spher.cor`, `spher.reg`

## Examples

```
x <- rnorm(100)
z <- cbind(3 + 2 * x, 1 -3 * x)
y <- cbind( rnorm(100,z[ ,1], 1), rnorm(100, z[ ,2], 1) )
y <- y / sqrt( rowSums(y^2) )
y <- ( atan( y[, 2] / y[, 1] ) + pi * I(y[, 1] < 0) ) %% (2 * pi)
a <- spml.reg(y, x, rads = TRUE, xnew = x)
b <- cipc.reg(y, x, rads = TRUE, xnew = x)
```

---

`Circular-linear correlation` *Circular-linear correlation*

---

## Description

It calculates the squared correlation between a circular and one or more linear variables.

## Usage

```
circlin.cor(theta, x, rads = FALSE)
```

## Arguments

| | |
|---|---|
| `theta` | The circular variable. |
| `x` | The linear variable or a matrix containing many linear variables. |
| `rads` | If the circualr variable is in rads, this should be TRUE and FALSE otherwise. |

## Details

The squared correlation between a circular and one or more linear variables is calculated.

## Value

A matrix with as many rows as linear variables including:

| | |
|---|---|
| `R-squared` | The value of the squared correlation. |
| `p-value` | The p-value of the zero correlation hypothesis testing. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris `<mtsagris@uoc.gr>` and Giorgos Athineou <gioathineou@gmail.com>.

## References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

`circ.cor1`, `circ.cor2`, `spml.reg`

## Examples

```
phi <- rvonmises(50, 2, 20, rads = TRUE)
x <- 2 * phi + rnorm(50)
y <- matrix(rnorm(50 * 5), ncol = 5)
circlin.cor(phi, x, rads = TRUE)
circlin.cor(phi, y, rads = TRUE)
```

---

# Column-wise MLE of the angular Gaussian and the von Mises Fisher distributions

*Column-wise MLE of the angular Gaussian and the von Mises Fisher distributions*

---

## Description

Column-wise MLE of the angular Gaussian and the von Mises Fisher distributions.

## Usage

```
colspml.mle(x ,tol = 1e-07, maxiters = 100, parallel = FALSE)
colvm.mle(x, tol = 1e-07)
```

## Arguments

| | |
|---|---|
| x | A numerical matrix with data. Each column refers to a different vector of observations of the same distribution. The values of for Lognormal must be greater than zero, for the logitnormal they must by percentages, exluding 0 and 1, whereas for the Borel distribution the x must contain integer values greater than 1. |
| tol | The tolerance value to terminate the Newton-Raphson algorithm. |
| maxiters | The maximum number of iterations that can take place in each regression. |
| parallel | Do you want this to be executed in parallel or not. The parallel takes place in C++, and the number of threads is defined by each system's availiable cores. |

## Details

For each column, spml.mle function is applied that fits the angular Gaussian distribution estimates its parameters and computes the maximum log-likelihood.

## Value

A matrix with four columns. The first two are the mean vector, then the $\gamma$ parameter, and the fourth column contains maximum log-likelihood.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Presnell Brett, Morrison Scott P. and Littell Ramon C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

## See Also

spml.mle, spml.reg, vmf.mle

## Examples

```
x <- matrix( runif(100 * 10), ncol = 10)
a <- colspml.mle(x)
b <- colvm.mle(x)
x <- NULL
```

---

# Column-wise uniformity Watson test for circular data

*Column-wise uniformity tests for circular data*

---

## Description

Column-wise uniformity tests for circular data.

## Usage

```
colwatsons(u, rads = FALSE)
```

## Arguments

| | |
|---|---|
| u | A numeric matrix containing the circular data which are expressed in radians. Each column is a different sample. |
| rads | A boolean variable. If the data are in radians, put this TRUE. If the data are expressed in degrees make this FALSE. |

## Details

These tests are used to test the hypothesis that the data come from a circular uniform distribution.

## Value

A matrix with two columns, the value of the test statistic and its associated p-value.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Jammalamadaka S. Rao and SenGupta A. (2001). Topics in Circular Statistics, pg. 156–157.

### See Also

watson, kuiper, fishkent

### Examples

```
x <- matrix( rvonmises(n = 50 * 10, m = 2, k = 0), ncol = 10 )
res<-colwatsons(x)
x <- NULL
```

---

## Contour plot (on the plane) of the ESAG and Kent distributions without any data

*Contour plot (on the plane) of the ESAG and Kent and ESAG distributions without any data*

---

### Description

The contour plot (on the plane) of the spherical ESAG and Kent distributions is produced.

### Usage

```
esag.contour(mu, gam, lat, long)
kent.contour(k, b)
```

### Arguments

| | |
|---|---|
| k | The concentration parameter. |
| b | The ovalness parameter. It has to be less than k/2 in order for the distribution to be unimodal. Otherwise it is bimodal. |
| mu | The mean vector the ESAG distribution, a vector in $R^3$. |
| gam | The two gamma parameters of the ESAG distribution. |
| lat | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| long | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

### Details

The goal of this function is for the user to see how the Kent or the SAG distribution looks like.

### Value

A plot containing the contours of the distribution.

### Author(s)

Michail Tsagris and Christos Adam.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Christos Adam <pada4m4@gmail.com>.

### References

Kent John (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society, Series B, 44(1): 71–80.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

### See Also

vmf.contour, vmf.kerncontour, spher.esag.contour

### Examples

```
kent.contour(10, 4)

mu <- colMeans( as.matrix( iris[,1:3] ) )
gam <- c(1,0.5)
esag.contour(mu, gam, 50, 50)
esag.contour(mu, gam, 30, 40)
```

---

## Contour plot (on the sphere) of a mixture of von Mises-Fisher distributions

*Contour plot (on the sphere) of a mixture of von Mises-Fisher distributions*

---

### Description

The contour plot (on the sphere) of a mixture of von Mises-Fisher distributions is produced.

### Usage

```
spher.mixvmf.contour(probs, mu, k, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
```

### Arguments

| | |
|---|---|
| probs | This is avector with the mixing probability of each group. |
| mu | A matrix with the mean direction of each group. |
| k | A vector with the concentration parameter of each group. |

| | |
|---|---|
| `bgcol` | The color of the surface of the sphere. |
| `dat` | If you have you want to plot supply them here. This has to be a numerical matrix with three columns, i.e. unit vectors. |
| `col` | If you supplied data then choose the color of the points. If you did not choose a color, the points will appear in red. |
| `lat` | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| `long` | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

## Details

The goal of this function is for the user to see how the mixtures of von Mises-Fisher look like.

## Value

A plot containing the contours of the mixture distribution.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Kurt Hornik and Bettina Grun (2014). movMF: An R Package for Fitting Mixtures of von Mises-Fisher Distributions http://cran.r-project.org/web/packages/movMF/vignettes/movMF.pdf

Mardia K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Sra S. (2012). A short note on parameter approximation for von Mises-Fisher distributions: and a fast implementation of $I_s(x)$. Computational Statistics, 27(1): 177–190.

## See Also

`spher.esag.contour`, `spher.vmf.contour`, `mixvmf.mle`

## Examples

```
k <- runif(3, 4, 20)
probs <- c(0.2, 0.5, 0.3)
mu <- matrix(rnorm(9, 0, 0.5), ncol = 3)
mu <- mu / sqrt( rowSums(mu^2) )
## the lat and long are decreased to 10. Increase them back to 50 to
## see the difference
spher.mixvmf.contour(probs, mu, k, lat = 10, long = 10)
```

---

`Contour plot (on the sphere) of some spherical rotationally symmetric distributions`

*Contour plot (on the sphere) of some spherical rotationally symmetric distributions*

---

### Description

The contour plot (on the sphere) of some spherical rotationally symmetric distributions is produced.

### Usage

```
spher.vmf.contour(mu, k, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
spher.purka.contour(theta, a, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
spher.spcauchy.contour(mu, rho, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
spher.pkbd.contour(mu, rho, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
```

### Arguments

| | |
|---|---|
| `mu` | The mean or the median direction, depending on the distribution, a unit vector. |
| `theta` | The mean direction (unit vector) of the Purkayastha distribution. |
| `k` | The concentration parameter ($\kappa$) of the von Mises-Fisher distribution. |
| `a` | The concentration parameter ($\alpha$) of the Purkayastha distribution. |
| `rho` | The concentration parameter ($\rho$) of the spherical Cauchy distribution. |
| `bgcol` | The color of the surface of the sphere. |
| `dat` | If you have you want to plot supply them here. This has to be a numerical matrix with three columns, i.e. unit vectors. |
| `col` | If you supplied data then choose the color of the points. If you did not choose a color, the points will appear in red. |
| `lat` | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| `long` | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

### Details

The goal of this function is for the user to see how the von Mises-Fisher, the Purkayastha, the spherical Cauchy or the Poisson kernel-based distribution looks like.

### Value

A plot containing the contours of the distribution.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Tsagris M., Papastamoulis P. and Kato S. (2024). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. https://arxiv.org/pdf/2409.03292.

Mardia K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Sra S. (2012). A short note on parameter approximation for von Mises-Fisher distributions: and a fast implementation of $I_s(x)$. Computational Statistics, 27(1): 177–190.

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83.

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics-Theory and Methods, 19(6): 1973–1986.

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248. https://arxiv.org/pdf/1510.07679.pdf

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Sablica L., Hornik K. and Leydold J. (2023). Efficient sampling from the PKBD distribution. Electronic Journal of Statistics, 17(2): 2180–2209.

### See Also

`spher.esag.contour`, `spher.mixvmf.contour`, `kent.contour`

### Examples

```
mu <- colMeans( as.matrix( iris[, 1:3] ) )
mu <- mu / sqrt( sum(mu^2) )
## the lat and long are decreased to 30. Increase them back to 50 to
## see the difference
spher.spcauchy.contour(mu, 0.7, lat = 30, long = 30)
```

---

## `Contour plot (on the sphere) of the ESAG and Kent distributions`

*Contour plot (on the sphere) of the ESAG and Kent distributions*

---

### Description

The contour plot (on the sphere) of the ESAG and Kent distributions is produced.

## Usage

```
spher.esag.contour(mu, gam, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
spher.kent.contour(G, param, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
```

## Arguments

| | |
|---|---|
| mu | The mean vector the ESAG distribution, a vector in $R^3$. |
| gam | The two gamma parameters of the ESAG distribution. |
| G | For the Kent distribution, a 3 x 3 matrix whose first column is the mean direction. The second and third columns are the major and minor axes respectively. |
| param | For the Kent distribution a vector with the concentration $\kappa$ and ovalness $\beta$ parameters. The angle $\psi$ has been absorbed inside the matrix G. |
| bgcol | The color of the surface of the sphere. |
| dat | If you have you want to plot supply them here. This has to be a numerical matrix with three columns, i.e. unit vectors. |
| col | If you supplied data then choose the color of the points. If you did not choose a color, the points will appear in red. |
| lat | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| long | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

## Details

The goal of this function is for the user to see how the ESAG or the Kent distribution looks like.

## Value

A plot containing the contours of the distribution.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Kent John (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society, Series B, 44(1): 71–80.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

## See Also

esag.contour, spher.purka.contour, kent.contour

### Examples

```
mu <- colMeans( as.matrix( iris[, 1:3] ) )
gam <- c(1 ,0.5)
## the lat and long are decreased to 30. Increase them back to 50 to
## see the difference
spher.esag.contour(mu, gam, lat = 30, long = 30)
```

---

`Contour plot (on the sphere) of the SESPC distribution`
*Contour plot (on the sphere) of the SESPC distribution*

---

### Description

The contour plot (on the sphere) of the SESPC distribution is produced.

### Usage

```
spher.sespc.contour(mu, theta, bgcol = "snow", dat = NULL, col = NULL,
lat = 50, long = 50)
```

### Arguments

| | |
|---|---|
| `mu` | The mean vector the SESPC distribution, a vector in $R^3$. |
| `theta` | The two $\theta$ parameters of the SESPC distribution. |
| `bgcol` | The color of the surface of the sphere. |
| `dat` | If you have you want to plot supply them here. This has to be a numerical matrix with three columns, i.e. unit vectors. |
| `col` | If you supplied data then choose the color of the points. If you did not choose a color, the points will appear in red. |
| `lat` | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| `long` | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

### Details

The goal of this function is for the user to see how the SESPC distribution looks like.

### Value

A plot containing the contours of the distribution.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

### See Also

spher.esag.contour, spher.spcauchy.contour

### Examples

```
mu <- colMeans( as.matrix( iris[, 1:3] ) )
theta <- c(1 ,0.5)
## the lat and long are decreased to 30. Increase them back to 50 to
## see the difference
spher.sespc.contour(mu, theta, lat = 30, long = 30)
```

---

## Contour plot of a mixture of von Mises-Fisher distributions model

*Contour plot of a mixture of von Mises-Fisher distributions model for spherical data only.*

---

### Description

Contour lines are produced of mixture model for spherical data only.

### Usage

```
mixvmf.contour(u, mod)
```

### Arguments

| | |
|---|---|
| u | A two column matrix. The first column is the longitude and the second is the latitude. |
| mod | This is mix.vmf object, actually it is a list. Run a mixture model and save it as mod for example, mod = mix.vmf(x, 3). |

### Details

The contour plot is displayed with latitude and longitude in the axes. No Lambert projection is used here. This works for spherical data only which are given as longitude and latitude.

### Value

A plot including: The points and the contour lines.

### Author(s)

Michail Tsagris and Christos Adam.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Christos Adam <pada4m4@gmail.com>.

### References

Kurt Hornik and Bettina Grun (2014). movMF: An R Package for Fitting Mixtures of von Mises-Fisher Distributions http://cran.r-project.org/web/packages/movMF/vignettes/movMF.pdf

### See Also

vmf.kerncontour, vmf.contour, mixvmf.mle

### Examples

```
k <- runif(2, 4, 20)
prob <- c(0.4, 0.6)
mu <- matrix( rnorm(6), ncol = 3 )
mu <- mu / sqrt( rowSums(mu^2) )
x <- rmixvmf(200, prob, mu, k)$x
mod <- mixvmf.mle(x, 2)
y <- euclid.inv(x)
mixvmf.contour(y, mod)
```

---

## Contour plot of spherical data using a von Mises-Fisher kernel density estimate

*Contour plot of spherical data using a von Mises-Fisher kernel density estimate*

---

### Description

Contour plot of spherical data using a von Mises-Fisher kernel density estimate.

### Usage

```
vmf.kerncontour(u, thumb = "none", den.ret = FALSE, full = FALSE, ngrid = 100)
```

### Arguments

| | |
|---|---|
| u | A two column matrix. The first coolumn is the latitude and the second is the longitude. |
| thumb | This is either 'none' (defualt), or 'rot' for the rule of thumb suggested by Garcia-Portugues (2013). If it is "none" it is estimated via cross validation, with the fast function vmfkde.tune. |

| | |
|---|---|
| `den.ret` | If FALSE (default), plots the contours of the density along with the individual points. If TRUE, will instead return a list with the Longitudes, Latitudes and Densities. Look at the 'value' section for details. |
| `full` | If FALSE (default), uses the range of positions from 'u' to calculate and optionally plot densities. If TRUE, calculates densities covering the entire sphere. |
| `ngrid` | Sets the resolution of the density calculation. |

## Details

It calculates the contour plot using a von Mises-Fisher kernel for spherical data only.

## Value

The contour lines of the data. If "den.ret" was set to TRUE a list including:

| | |
|---|---|
| `lat` | The latitude values. |
| `long` | The longitude values. |
| `h` | The optimal bandwidth. |
| `den` | The kernel density estimate contour points. |

## Author(s)

Michail Tsagris, Micah J. Waldstein and Christos Adam.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>, Micah J. Waldstein <micah@waldste.in> and Christos Adam <pada4m4@gmail.com>.

## References

Garcia Portugues, E. (2013). Exact risk improvement of bandwidth selectors for kernel density estimation with directional data. Electronic Journal of Statistics, 7, 1655–1685.

## See Also

vmf.kde, vmfkde.tune, vmf.contour

## Examples

```
x <- rvmf(100, rnorm(3), 15)
x <- euclid.inv(x)

vmf.kerncontour(x, "rot")
```

Contour plots of some rotationally symmetric distributions
*Contour plots of some rotationally symmetric distributions*

## Description

Contour plots of some rotationally symmetric distributions.

## Usage

```
vmf.contour(k)
spcauchy.contour(mu, rho, lat = 50, long = 50)
purka.contour(theta, a, lat = 50, long = 50)
pkbd.contour(mu, rho, lat = 50, long = 50)
```

## Arguments

| | |
|---|---|
| `k` | The concentration parameter. |
| `mu` | The mean direction (unit vector) of the von Mises-Fisher, the IAG, the spherical Cauchy distribution, or the Poisson kernel-based distribution. |
| `rho` | The $\rho$ parameter of the spherical Cauchy distribution, or the Poisson kernel-based distribution. |
| `theta` | The median direction for the Purkayastha distribution, a unit vector. |
| `a` | The concentration parameter of the Purkayastha distribution. |
| `lat` | A positive number determing the range of degrees to move left and right from the latitude center. See the example to better understand this argument. |
| `long` | A positive number determing the range of degrees to move up and down from the longitude center. See the example to better understand this argument. |

## Details

The user specifies the concentration parameter only and not the mean direction or data. This is for illustration purposes only. The graph of the von Mises-Fisher distribution will always contain circles, as this distribution is the analogue of a bivariate normal in two dimensions with a zero covariance.

## Value

A contour plot of the distribution.

## Author(s)

Michail Tsagris and Christos Adam.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Christos Adam <pada4m4@gmail.com>.

### References

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248. https://arxiv.org/pdf/1510.07679.pdf

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics-Theory and Methods, 19(6): 1973–1986.

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Sablica L., Hornik K. and Leydold J. (2023). Efficient sampling from the PKBD distribution. Electronic Journal of Statistics, 17(2): 2180–2209.

### See Also

rvmf, vmf.mle, vmf.kerncontour, kent.contour, sphereplot

### Examples

```
vmf.contour(5)
mu <- colMeans( as.matrix( iris[,1:3] ) )
mu <- mu / sqrt( sum(mu^2) )
spcauchy.contour(mu, 0.7, 30, 30)
spcauchy.contour(mu, 0.7, 60, 60)
```

---

## Conversion of cosines to azimuth and plunge

*Conversion of cosines to azimuth and plunge*

---

### Description

Conversion of cosines to azimuth and plunge.

### Usage

```
cosap(x,y,z)
```

### Arguments

| | |
|---|---|
| x | x component of cosine. |
| y | y component of cosine. |
| z | z component of cosine. |

### Details

Orientation: x>0 is 'eastward', y>0 is 'southward', and z>0 is 'downward'.

### Value

A list including:

`A` The azimuth

`P` The plunge

### Author(s)

Eli Amson.

R implementation and documentation: Eli Amson <eli.amson1988@gmail.com>.

### References

Amson E, Arnold P, Van Heteren AH, Cannoville A, Nyakatura JA. Trabecular architecture in the forelimb epiphyses of extant xenarthrans (Mammalia). Frontiers in Zoology.

### See Also

`euclid`, `euclid.inv`, `eul2rot`

### Examples

```
cosap(-0.505, 0.510, -0.696)
```

---

`Converting a rotation matrix on SO(3) to an unsigned unit quaternion`
*Converting a rotation matrix on SO(3) to an unsigned unit quaternion*

---

### Description

It returns an unsigned unite quaternion in $S^3$ (the four-dimensional sphere) from a $3 \times 3$ rotation matrix on SO(3).

### Usage

```
rot2quat(X)
```

### Arguments

`X` A rotation matrix in SO(3).

## Details

Firstly construct a system of linear equations by equating the corresponding components of the theoretical rotation matrix proposed by Prentice (1986), and given a rotation matrix. Finally, the system of linear equations are solved by following the tricks mentioned in second reference here in order to achieve numerical accuracy to get quaternion values.

## Value

A unsigned unite quaternion.

## Author(s)

Anamul Sajib.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

## References

Prentice,M. J. (1986). Orientation statistics without parametric assumptions.Journal of the Royal Statistical Society. Series B: Methodological 48(2). //http://www.euclideanspace.com/maths/geometry/rotations/conversions/

## See Also

quat2rot, rotation, Arotation \ link{rot.matrix}

## Examples

```
x <- rnorm(4)
x <- x/sqrt( sum(x^2) ) ## an unit quaternion in R4 ##
R <- quat2rot(x)
R
x
rot2quat(R) ## sign is not exact as you can see
```

---

# Converting an unsigned unit quaternion to rotation matrix on SO(3)

*Converting an unsigned unit quaternion to rotation matrix on SO(3)*

---

## Description

It forms a (3 x 3) rotation matrix on SO(3) from an unsigned unite quaternion in $S^3$ (the four-dimensional sphere).

## Usage

```
quat2rot(x)
```

### Arguments

x    An unsigned unit quaternion in $S^3$.

### Details

Given an unsigned unit quaternion in $S^3$ it forms a rotation matrix on SO(3), according to the transformation proposed by Prentice (1986).

### Value

A rotation matrix.

### Author(s)

Anamul Sajib.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

### References

Prentice,M. J. (1986). Orientation statistics without parametric assumptions.Journal of the Royal Statistical Society. Series B: Methodological 48(2).

### See Also

`rot2quat`, `rotation`, `Arotation` `rot.matrix`

### Examples

```
x <- rnorm(4)
x <- x/sqrt( sum(x^2) )
x                    ## an unit quaternion in R4 ##
quat2rot(x)
```

---

## Cross validation for estimating the classification rate

*Cross validation for estimating the classification rate*

---

### Description

Cross validation for estimating the classification rate.

### Usage

```
dirda.cv(x, ina, folds = NULL, nfolds = 10, stratified = FALSE,
         type = c("vmf", "iag", "esag", "kent", "sc", "pkbd", "purka"),
         seed = NULL, B = 1000)
```

### Arguments

| | |
|---|---|
| x | A matrix with the data in Eulcidean coordinates, i.e. unit vectors. The matrix must have three columns, only spherical data are currently supported. |
| ina | A variable indicating the groupings. |
| folds | Do you already have a list with the folds? If not, leave this NULL. |
| nfolds | How many folds to create? |
| stratified | Should the folds be created in a stratified way? i.e. keeping the distribution of the groups similar through all folds? |
| seed | If seed is TRUE, the results will always be the same. |
| type | The type of classifier to use. The avaliable options are "vmf" (von Mises-Fisher distribution), "iag" (IAG distribution), "esag" (ESAG distribution), "kent" (Kent distribution), "sc" and "sc2" (spherical Cauchy distribution), "pkbd" and "pkbd2" (Poisson kernel-based distribution), and "purka" (Purkayastha distribution). The difference between "sc" and "sc2" and between "pkbd" and "pkbd2" is that the first uses the Newton-Raphson algorithm and it is faster, whereas the second uses a hybrid algorithm that does not require the Hessian matrix, but in large dimensions the second will be faster. You can chose any of them or all of them. Note that "kent" works only with spherical data. |
| B | If you used k-NN, should a bootstrap correction of the bias be applied? If yes, 1000 is a good value. |

### Details

Cross-validation for the estimation of the performance of a classifier.

The estimated performance of the best classifier is overestimated. After the cross-valdiation procedure, the predicted values produced by all classifiers are colelcted, from all folds, in an $n \times M$ matrix, where $n$ is the number of samples and $M$ is the number of all classifiers used. We sample rows (predictions) with replacement from P and denote them as the in-sample values. The non re-sampled rows are denoted as out-of-sample values. The performance of each classifier in the insample rows is calculated and the classifier with the optimal performance is selected, followed by the calculation of performance in the out-of-sample values. This process is repeated B times and the average performance is returned. The only computational overhead is with the repetitive resampling and calculation of the performance, i.e. no model or classifier is fitted nor trained. For more information see Tsamardinos et al. (2018).

The good thing with the function is that you can run any method you want by supplying the folds yourselves using the command `makefolds`. Then suppose you want to run another method. By suppying the same folds you will be able to have comparative results for all methods.

### Value

A list including:

| | |
|---|---|
| perf | A vector with the estimated performance of each classifier. |
| bbc.perf | The bootstrap bias corrected performance. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

Tsagris M. and Alenazi A. (2019). Comparison of discriminant analysis methods on the sphere. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(4), 467–491.

Mardia K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Morris J. E. and Laycock P. J. (1974). Discriminant analysis of directional data. Biometrika, 61(2): 335–341.

Tsamardinos I., Greasidou E. and Borboudakis G. (2018). Machince Learning, 107(12): 1895–1922.

### See Also

dirda, dirknn, knn.reg

### Examples

```
x <- rvmf(300, rnorm(3), 10)
ina <- sample.int(2, 300, replace = TRUE)
dirda.cv(x, ina, B = 1)
```

---

## Cumulative distribution function of circular distributions

*Cumulative distribution function of circular distributions*

---

### Description

Cumulative probability distribution of circular distributions.

### Usage

```
pvm(u, m, k, rads = FALSE)
pspml(u, mu, rads = FALSE)
pwrapcauchy(u, m, rho, rads = FALSE)
pcircpurka(u, m, a, rads = FALSE)
pcircbeta(u, m, a, b, rads = FALSE)
pcardio(u, m, rho, rads = FALSE)
pcircexp(u, lambda, rads = FALSE)
pcipc(u, omega, g, rads = FALSE)
pgcpc(u, omega, g, rho, rads = FALSE)
pmmvm(u, m, k, N, rads = FALSE)
```

### Arguments

| | |
|---|---|
| `u` | A numerical value, either in radians or in degrees. |
| `m` | The mean direction of the von Mises and the multi-modal von Mises distribution in radians or in degrees. |
| `mu` | The mean vector, a vector with two values for the "pspml". |
| `omega` | The location parameter of the CIPC and GCPC distributions. |
| `g` | The norm of the mean vector for the CIPC and GCPC distributions. |
| `k` | The concentration parameter, $\kappa$. |
| `lambda` | The $\lambda$ parameter of the circular exponential distribution. This must be positive. |
| `a` | The $\alpha$ parameter of the circular Purkayastha distribution or the $\alpha$ parameter of the circular Beta distribution. |
| `b` | The $\beta$ parameter of the circular beta distribution. |
| `rho` | The $\rho$ parameter of the Cardioid, wrapped Cauchy and GCPC distributions. |
| `N` | The number of modes to consider in the multi-modal von Mises distribution. |
| `rads` | If the data are in radians, this should be TRUE and FALSE otherwise. |

### Details

This value calculates the probability of u being less than some value $\theta$.

### Value

The probability that of u being less than $\theta$, where u follows a circular distribution.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Arthur Pewsey, Markus Neuhauser, and Graeme D. Ruxton (2013). Circular Statistics in R.

Barnett M. J. and Kingston R. L. (2024). A note on the Hendrickson-Lattman phase probability distribution and its equivalence to the generalized von Mises distribution. Journal of Applied Crystallography, 57(2).

Jammalamadaka S. R. and Kozubowski T. J. (2003). A new family of circular models: The wrapped Laplace distributions. Advances and Applications in Statistics, 3(1): 77–103.

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics–Theory and Methods, 19(6): 1973–1986.

Paula F. V., Nascimento A. D., Amaral G. J. and Cordeiro G. M. (2021). Generalized Cardioid distributions for circular data analysis. Stats, 4(3): 634–649.

Zheng Sun (2009). Comparing measures of fit for circular distributions. MSc Thesis, University of Victoria. file:///C:/Users/mtsag/Downloads/zhengsun_master_thesis.pdf

### See Also

group.gof, dvm, dcircexp,purka.mle, dcircpurka, dmmvm

### Examples

```
pvm(1, 2, 10, rads = TRUE)
pmmvm(1, 2, 10, 3, rads = TRUE)
pcircexp(c(1, 2), 2, rads = TRUE)
pcircpurka(2, 3, 0.3)
```

---

## Density of a mixture of rotationally symmetric distributions

*Density of a mixture of rotationally symmetric distributions*

---

### Description

Density of a mixture of rotationally symmetric distributions.

### Usage

```
dmixvmf(y, probs, mu, k, logden = FALSE)
dmixspcauchy(y, probs, mu, rho, logden = FALSE)
dmixpkbd(y, probs, mu, rho, logden = FALSE)
```

### Arguments

| | |
|---|---|
| y | A matrix with unit vectors. |
| probs | This is avector with the mixing probability of each group. |
| mu | A matrix with the mean direction of each group. |
| k | A vector with the concentration parameter of each group. |
| rho | A vector with the concentration parameter of each group. |
| logden | If you the logarithm of the density values set this to TRUE. |

### Details

The function computes the density for a given mixture of von Mises-Fisher, spherical Cauchy or Poisson kernel-based distributions.

### Value

A vector with the (log) density values of y.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Kurt Hornik and Bettina Grun (2014). movMF: An R Package for Fitting Mixtures of von Mises-Fisher Distributions http://cran.r-project.org/web/packages/movMF/vignettes/movMF.pdf

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

## See Also

mixvmf.mle, rvmf, bic.mixvmf

## Examples

```
k <- runif(3, 4, 6)
probs <- c(0.2, 0.5, 0.3)
mu <- matrix(rnorm(9), ncol = 3)
mu <- mu / sqrt( rowSums(mu^2) )
x <- rmixvmf(200, probs, mu, k)$x
b <- dmixvmf(x, probs, mu, k)
```

---

Density of some (hyper-)spherical distributions
*Density of some (hyper-)spherical distributions*

---

## Description

Density of some (hyper-)spherical distributions.

## Usage

```
dvmf(y, mu, k, logden = FALSE )
iagd(y, mu, logden = FALSE)
dpurka(y, theta, a, logden = FALSE)
dspcauchy(y, mu, rho, logden = FALSE)
dpkbd(y, mu, rho, logden = FALSE)
```

## Arguments

| | |
|---|---|
| y | A matrix or a vector with the data expressed in Euclidean coordinates, i.e. unit vectors. |
| mu | The mean direction (unit vector) of the von Mises-Fisher, the IAG, the spherical Cauchy distribution, or of the Poisson kernel-based distribution. |
| theta | The mean direction (unit vector) of the Purkayastha distribution. |
| k | The concentration parameter of the von Mises-Fisher distribution. |
| a | The concentration parameter of the Purkayastha distribution. |
| rho | The $\rho$ parameter of the spherical Cauchy distribution, or of the Poisson kernel-based distribution. |
| logden | If you the logarithm of the density values set this to TRUE. |

## Details

The density of the von Mises-Fisher, of the IAG, of the Purkayastha, of the spherical Cauchy distribution, or of the Poisson kernel-based distribution is computed.

## Value

A vector with the (log) density values of y.

## Author(s)

Michail Tsagris and Zehao Yu.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Zehao Yu <zehaoy@email.sc.edu>.

## References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics-Theory and Methods, 19(6): 1973–1986.

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248. https://arxiv.org/pdf/1510.07679.pdf

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Sablica L., Hornik K. and Leydold J. (2023). Efficient sampling from the PKBD distribution. Electronic Journal of Statistics, 17(2): 2180–2209.

Zehao Yu and Xianzheng Huang (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statististics, 18(1): 301–334.

Tsagris M., Papastamoulis P. and Kato S. (2024). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. https://arxiv.org/pdf/2409.03292.

## See Also

kent.mle, rkent, esag.mle

## Examples

```
m <- colMeans( as.matrix( iris[,1:3] ) )
y <- rvmf(1000, m = m, k = 10)
dvmf(y, k=10, m)
```

---

Density of some circular distributions *Density of some circular distributions*

---

## Description

Density of some circular distributions.

## Usage

```
dvm(x, m, k, rads = FALSE, logden = FALSE)
dspml(x, mu, rads = FALSE, logden = FALSE)
dwrapcauchy(x, m, rho, rads = FALSE, logden = FALSE)
dwrapnormal(x, m, rho, rads = FALSE, logden = FALSE)
dcircpurka(x, m, a, rads = FALSE, logden = FALSE)
dggvm(x, param, rads = FALSE, logden = FALSE)
dcircbeta(x, m, a, b, rads = FALSE, logden = FALSE)
dcardio(x, m, rho, rads = FALSE, logden = FALSE)
dcircexp(x, lambda, rads = FALSE, logden = FALSE)
dcipc(x, omega, g, rads = FALSE, logden = FALSE)
dgcpc(x, omega, g, rho, rads = FALSE, logden = FALSE)
dmmvm(x, m, k, N, rads = FALSE, logden = FALSE)
```

## Arguments

| | |
|---|---|
| `x` | A vector with circular data. |
| `m` | The mean value of the von Mises, wrapped Cauchy, wrapped normal and of the cardioid distribution, a scalar. This is the median for the circular Purkayastha distribution. |
| `mu` | The mean vector, a vector with two values for the "spml" and the GCPC. |
| `omega` | The location parameter of the CIPC and GCPC distributions. |
| `g` | The norm of the mean vector for the CIPC and GCPC distributions. |
| `k` | The concentration parameter. |
| `rho` | For the wrapped Cauchy, normal and Cardioid distributions, this is the $\rho$ parameter. For the GCPC distribution this is the eigenvalue parameter, or covariance determinant. |
| `a` | The $\alpha$ parameter of the circular Purkayastha distribution or the $\alpha$ parameter of the circular Beta distribution. |
| `b` | The $\beta$ parameter of the circular Beta distribution. |
| `lambda` | The $\lambda$ parameter of the circular (or wrapped) exponential distribution. This must be positive. |
| `param` | The vector of parameters of the GGVM distribution as returned by the function `ggvm.mle`. |
| `N` | The number of modes to consider in the multi-modal von Mises distribution. |

| | |
|---|---|
| `rads` | If the data are in rads, then this should be TRUE, otherwise FALSE. |
| `logden` | If you the logarithm of the density values set this to TRUE. |

## Details

The density of the von Mises, bivariate projected normal, cardio, circular exponential, wrapped Cauchy, wrapped normal, circular Purkayastha, CIPC or GCPC distributions is computed.

## Value

A vector with the (log) density values of x.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Presnell B., Morrison S. P. and Littell R. C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

Jammalamadaka S. R. and Kozubowski T. J. (2003). A new family of circular models: The wrapped Laplace distributions. Advances and Applications in Statistics, 3(1): 77–103.

Barnett M. J. and Kingston R. L. (2024). A note on the Hendrickson-Lattman phase probability distribution and its equivalence to the generalized von Mises distribution. Journal of Applied Crystallography, 57(2).

Paula F. V., Nascimento A. D., Amaral G. J. and Cordeiro G. M. (2021). Generalized Cardioid distributions for circular data analysis. Stats, 4(3): 634–649.

Zheng Sun (2009). Comparing measures of fit for circular distributions. MSc Thesis, University of Victoria. file:///C:/Users/mtsag/Downloads/zhengsun_master_thesis.pdf

Lopez-Custodio P. C. (2024). A cheat sheet for probability distributions of orientational data. arXiv:2412.08934.

## See Also

dkent, rvonmises, desag

## Examples

```
x <- rvonmises(500, m = 2.5, k = 10, rads = TRUE)
mod <- circ.summary(x, rads = TRUE, plot = FALSE)
den <- dvm(x, mod$mesos, mod$kappa, rads = TRUE, logden = TRUE )
mod$loglik
sum(den)
```

---

Density of the SESPC distribution

*Density of the SESPC distribution*

---

## Description

Density of the SESPC distribution.

## Usage

```
dsespc(y, mu, theta, logden = FALSE)
```

## Arguments

| | |
|---|---|
| `y` | A matrix or a vector with the data expressed in Euclidean coordinates, i.e. unit vectors. |
| `mu` | The mean vector the SESPC distribution, a vector in $R^3$. |
| `theta` | The two $\theta$ parameters of the SESPC distribution. |
| `logden` | If you the logarithm of the density values set this to TRUE. |

## Details

The density of the SESPC distribution is computed.

## Value

A vector with the (log) density values of y.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

`desag`, `sespc.mle`

## Examples

```
m <- colMeans( as.matrix( iris[,1:3] ) )
y <- rsespc(1000, m, c(1, 1))
mod <- sespc.mle(y)
dsespc( y, mod$mu, mod$theta)
```

---

# Density of the spherical ESAG and Kent distributions and of the ESAG distribution in arbitrary dimensions

*Density of the spherical ESAG and Kent distributions*

---

## Description

Density of the spherical ESAG and Kent distributions.

## Usage

```
desag(y, mu, gam, logden = FALSE)
dkent(y, G, param, logden = FALSE)
dESAGd(y, mu, gam, logden = FALSE)
```

## Arguments

| | |
|---|---|
| `y` | A matrix or a vector with the data expressed in Euclidean coordinates, i.e. unit vectors. For the dESAGd it can have any dimension. |
| `mu` | The mean vector the ESAG distribution. |
| `gam` | The two $\gamma$ parameters of the ESAG distribution. |
| `G` | For the Kent distribution only, a 3 x 3 matrix whose first column is the mean direction. The second and third columns are the major and minor axes respectively. |
| `param` | For the Kent distribution a vector with the concentration $\kappa$ and ovalness $\beta$ parameters. The $\psi$ has been absorbed inside the matrix G. |
| `logden` | If you the logarithm of the density values set this to TRUE. |

## Details

The density of the spherical ESAG or Kent distribution, or of the ESAG distribution in arbitrary dimensions is computed.

## Value

A vector with the (log) density values of y.

## Author(s)

Michail Tsagris and Zehao Yu.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Zehao Yu <Zzehaoy@email.sc.edu>.

### References


Zehao Yu and Xianzheng Huang (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statististics, 18(1): 301–334.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

Kent John (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society, Series B, 44(1): 71–80.

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.


### See Also

kent.mle, rkent, esag.mle

### Examples

```
m <- colMeans( as.matrix( iris[, 1:3] ) )
y <- rkent(1000, k = 10, m = m, b = 4)
mod <- kent.mle(y)
dkent( y, G = mod$G, param = mod$param )
```

---

## Density of the Wood bimodal distribution on the sphere

*Density of the Wood bimodal distribution on the sphere*

---

### Description

Density of the Wood bimodal distribution on the sphere.

### Usage

```
dwood(y, param, logden = FALSE)
```

### Arguments

| | |
|---|---|
| y | A matrix containing two columns. The first one is the latitude and the second is the longitude, both expressed in degrees. |
| param | A vector with the 5 parameters, in the order they are returned by the wood.mle function. That is, $(\gamma, \delta, \alpha, \beta, \kappa)$. |
| logden | If you the logarithm of the density values set this to TRUE. |

### Details

The density of the spherical Wood distribution is computed.

### Value

A vector with the (log) density values of y.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Wood A.T.A. (1982). A bimodal distribution on the sphere. Journal of the Royal Statistical Society, Series C, 31(1): 52–58.

### See Also

dkent, desag, wood.mle

### Examples

```
x <- rvmf(100, rnorm(3), 15)
x <- euclid.inv(x)
mod <- wood.mle(x)
d <- dwood(x, mod$info[, 1])
```

---

## Euclidean transformation

*Euclidean transformation*

---

### Description

It transforms the data from the spherical coordinates to Euclidean coordinates.

### Usage

```
euclid(u)
```

### Arguments

| | |
|---|---|
| u | A two column matrix or even one single vector, where the first column (or element) is the latitude and the second is the longitude. The order is important. |

### Details

It takes the matrix of unit vectors of latitude and longitude and transforms it to unit vectors.

### Value

A three column matrix:

| | |
|---|---|
| U | The Euclidean coordinates of the latitude and longitude. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### See Also

euclid.inv, Arotation, lambert

### Examples

```
x <- rvmf(10, rnorm(3), 10)
u <- euclid.inv(x)
euclid(u)
x
```

---

Euler angles from a rotation matrix on SO(3)
*Compute the Euler angles from a rotation matrix on SO(3).*

---

### Description

It calculates three euler angles $(\theta_{12}, \theta_{13}, \theta_{23})$ from a $(3 \times 3)$ rotation matrix X, where X is defined as $X = R_z(\theta_{12}) \times R_y(\theta_{13}) \times R_x(\theta_{23})$. Here $R_x(\theta_{23})$ means a rotation of $\theta_{23}$ radians about the x axis.

### Usage

```
rot2eul(X)
```

### Arguments

| | |
|---|---|
| X | A rotation matrix which is defined as a product of three elementary rotations mentioned above. Here $\theta_{12}, \theta_{23} \in (-\pi, \pi)$ and and $\theta_{13} \in (-\pi/2, \pi/2)$. |

### Details

Given a rotation matrix X, euler angles are computed by equating each element in X with the corresponding element in the matrix product defined above. This results in nine equations that can be used to find the euler angles.

### Value

For a given rotation matrix, there are two eqivalent sets of euler angles.

### Author(s)

Anamul Sajib <sajibstat@du.ac.bd>.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

### References

Green, P. J. and Mardia, K. V. (2006). Bayesian alignment using hierarchical models, with applications in proteins bioinformatics. Biometrika, 93(2):235–254.

http://www.staff.city.ac.uk/~sbbh653/publications/euler.pdf

### See Also

eul2rot

### Examples

```
# three euler angles

theta.12 <- sample( seq(-3, 3, 0.3), 1 )
theta.23 <- sample( seq(-3, 3, 0.3), 1 )
theta.13 <- sample( seq(-1.4, 1.4, 0.3), 1 )

theta.12 ; theta.23 ; theta.13

X <- eul2rot(theta.12, theta.23, theta.13)
X  ##  A rotation matrix

e <- rot2eul(X)$v1

theta.12 <- e[3]
theta.23 <- e[2]
theta.13 <- e[1]

theta.12 ; theta.23 ; theta.13
```

---

## Forward Backward Early Dropping selection for circular data using the SPML regression

*Forward Backward Early Dropping selection for circular data using the SPML regression*

---

### Description

Forward Backward Early Dropping selection for circular data using the SPML regression.

### Usage

```
spml.fbed(y, x, alpha = 0.05, K = 0, backward = FALSE,
         parallel = FALSE, tol = 1e-07, maxiters = 100)
```

### Arguments

| | |
|---|---|
| `y` | The response variable, a numeric vector expressed in rads. |
| `x` | A matrix with continuous independent variables. |
| `alpha` | The significance threshold value for assessing p-values. Default value is 0.05. |
| `K` | How many times should the process be repeated? The default value is 0. |
| `backward` | After the Forward Early Dropping phase, the algorithm proceeds witha the usual Backward Selection phase. The default value is set to TRUE. It is advised to perform this step as maybe some variables are false positives, they were wrongly selected. This is rather experimental now and there could be some mistakes in the indices of the selected variables. **Do not use it for now**. |
| `parallel` | If you want the algorithm to run in parallel set this TRUE. |
| `tol` | The tolerance value to terminate the Newton-Raphson algorithm. |
| `maxiters` | The maximum number of iterations Newton-Raphson will perform. |

### Details

The algorithm is a variation of the usual forward selection. At every step, the most significant variable enters the selected variables set. In addition, only the significant variables stay and are further examined. The non signifcant ones are dropped. This goes until no variable can enter the set. The user has the option to re-do this step 1 or more times (the argument K). In the end, a backward selection is performed to remove falsely selected variables. Note that you may have specified, for example, K=10, but the maximum value FBED used can be 4 for example.

### Value

If K is a single number a list including: Note, that the "gam" argument must be the same though.

| | |
|---|---|
| `res` | A matrix with the selected variables and their test statistic. |
| `info` | A matrix with the number of variables and the number of tests performed (or models fitted) at each round (value of K). This refers to the forward phase only. |
| `runtime` | The runtime required. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Borboudakis G. and Tsamardinos I. (2019). Forward-backward selection with early dropping. Journal of Machine Learning Research, 20(8): 1–39.

Tsagis M. (2018). Guide on performing feature selection with the R package MXM. https://f1000research.com/articles/7-1505

Presnell Brett, Morrison Scott P. and Littell Ramon C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

## See Also

spml.reg, spml.regs, spml.mle

## Examples

```
x <- matrix( runif(100 * 50, 1, 100), ncol = 50 )
y <- runif(100)
a <- spml.fbed(y, x)
```

---

Generate random folds for cross-validation

*Generate random folds for cross-validation*

---

## Description

Random folds for use in a cross validation are generated. There is the option for stratified splitting as well.

## Usage

```
makefolds(ina, nfolds = 10, stratified = TRUE, seed = NULL)
```

## Arguments

| | |
|---|---|
| ina | A variable indicating the groupings. |
| nfolds | The number of folds to produce. |
| stratified | A boolean variable specifying whether stratified random (TRUE) or simple random (FALSE) sampling is to be used when producing the folds. |
| seed | You can specify your own seed number here or leave it NULL. |

## Details

I was inspired by the command in the package **TunePareto** in order to do the stratified version.

## Value

A list with nfolds elements where each elements is a fold containing the indices of the data.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## See Also

dirda.cv

## Examples

```
a <- makefolds(iris[, 5], nfolds = 5, stratified = TRUE)
table(iris[a[[1]], 5])  ## 10 values from each group
```

---

Generation of unit vector(s) with a given angle
*Generation of unit vector(s) with a given angle*

---

## Description

Generation of unit vector(s) with a given angle from a given unit vector.

## Usage

```
vec(x, n = 1, deg = 90)
```

## Arguments

| | |
|---|---|
| x | A unit vector. If it is not a unit vector it becomes one. |
| n | The number of unit vectors to return. |
| deg | The angle between the given vector and the n vectors to be returned. This must be in degrees and it has to be between 0 and 180 degrees. If the angle is 0, the same unit vector will be returned. If the angle is 180, the same unit vector with the signs changed will be returned. |

## Details

The user provides a unit vector and the degrees. The function will return n unit vectors whose angle with the given unit vector equals the degrees given. For example, if you want 10 unit vectors purpendicualr to the x put vec(x, 10, 90).

## Value

A list including:

| | |
|---|---|
| runtime | The runtime of the procedure. |
| crit | The calculated angle between the given unit vector and each of the generated unit vectors. |
| mat | A matrix with the n unit vectors. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### See Also

rvmf, rbingham, rfb

### Examples

```
x <- rnorm(10)
x <- x / sqrt( sum(x^2) )
a <- vec(x, 20, 90)
```

---

## Goodness of fit test for grouped data

*Goodness of fit test for grouped data*

---

### Description

Goodness of fit test for grouped data.

### Usage

```
group.gof(g, ni, m, k, dist = "vm", rads = FALSE, R = 999, ncores = 1)
```

### Arguments

| | |
|---|---|
| g | A vector with the group points, either in radians or in degrees. |
| ni | The frequency of each or group class. |
| m | The mean direction in radians or in degrees. |
| k | The concentration parameter, $\kappa$. |
| dist | The distribution to be tested, it can be either "vm" or "uniform". |
| rads | If the data are in radians, this should be TRUE and FALSE otherwise. |
| R | The number of bootstrap simulations to perform, set to 999 by default. |
| ncores | The number of cores to use. |

### Details

When you have grouped data, you can test whether the data come from the von Mises-Fisher distribution or from a uniform distribution.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| statistic | The test statistic value. |
| parameter | Since this is a bootstrap based test, there are no degrees of freedom, hence this is "NA". |
| p.value | The p-value of the test. |

| | |
|---|---|
| alternative | A character with the alternative hypothesis. |
| method | A character with the test used. |
| data.name | A character vector with two elements. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Arthur Pewsey, Markus Neuhauser, and Graeme D. Ruxton (2013). Circular Statistics in R.

## See Also

pvm, circ.summary, rvonmises

## Examples

```
x <- rvonmises(100, 2, 10)
g <- seq(min(x) - 0.1, max(x) + 0.1, length = 6)
ni <- as.vector( table( cut(x, g) ) )
group.gof(g, ni, 2, 10, dist = "vm", rads = TRUE, R = 299, ncores = 1)
group.gof(g, ni, 2, 5, dist = "vm", rads = TRUE, R = 299, ncores = 1)
```

---

# Habeck's rotation matrix generation

*Generation of three-dimensional random rotations using Habeck's algorithm.*

---

## Description

It generates random rotations in three-dimensional space that follow a probability distribution, matrix Fisher distribution, arising in fitting and matching problem.

## Usage

```
habeck.rot(F)
```

## Arguments

| | |
|---|---|
| F | An arbitrary 3 x 3 matrix represents the parameter matrix of this distribution. |

## Details

Firstly rotation matrices **X** are chosen which are the closest to F, and then parameterized using euler angles. Then a Gibbs sampling algorithm is implemented to generate rotation matrices from the resulting disribution of the euler angles.

## Value

A simulated rotation matrix.

## Author(s)

Anamul Sajib.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

## References

Habeck M (2009). Generation of three-dimensional random rotations in fitting and matching problems. Computational Statistics, 24, 719–731.

## Examples

```
F <- 10^(-1) * matrix( c(85, 11, 41, 78, 39, 60, 43, 64, 48), ncol = 3 ) ## Arbitrary F matrix
X <- habeck.rot(F)
det(X)
```

---

`Haversine distance matrix`

# *Harvesine distance matrix*

---

## Description

Haversine distance matrix.

## Usage

```
haversine.dist(x)
```

## Arguments

| | |
|---|---|
| `x` | A a matrix of two columns. The first column is the latitude and the second the longitude. |

## Details

The function computes the haversine distance between all observations.

## Value

A matrix with the haversine distances between all observations.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

https://en.wikipedia.org/wiki/Haversine_formula

### See Also

cosnn, dirknn

### Examples

```
x <- rvmf(10, rnorm(3), 10)
x <- euclid.inv(x)
haversine.dist(x)
```

---

Hyper spherical-spherical regression

## *Hyper spherical-spherical regression*

---

### Description

Regression when both the dependent and independent variables are directional data-.

### Usage

```
hspher.reg(y, x, xnew = NULL)
```

### Arguments

| | |
|---|---|
| y | The dependent variable; a matrix with either two columns, latitude and longitude, either in radians or in degrees. Alternatively it is a matrix with three columns, unit vectors. |
| x | The dependent variable; a matrix with either two columns, latitude and longitude, either in radians or in degrees. Alternatively it is a matrix with three columns, unit vectors. The two matrices must agree in the scale and dimensions. |
| xnew | The new values of some directional independent variable(s) whose directional response values you want to predict. If you have no new x values, leave it NULL (default). |

### Details

Spherical regression as proposed by Chang (1986) is implemented. If the estimated rotation matrix has a determinant equal to -1, singular value decomposition is performed and the last unit vector is multiplied by -1.

## Value

A list including:

| | |
|---|---|
| A | The estimated rotation matrix. |
| est | The fitted values in unit vectors, if the argument xnew is not NULL. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Ted Chang (1986). Spherical Regression. Annals of Statistics, 14(3): 907–924.

## See Also

spher.cor, spml.reg, spher.reg, sphereplot

## Examples

```
mx <- rnorm(5)
mx <- mx/sqrt( sum(mx^2) )
my <- rnorm(5)
my <- my/sqrt( sum(my^2) )
x <- rvmf(100, mx, 15)
A <- rotation(mx, my)
y <- x %*% t(A)
mod <- hspher.reg(y, x)
A
mod$A ## exact match, no noise
y <- x %*% t(A)
y <- y + rvmf(100, colMeans(y), 40)
mod <- hspher.reg(y, x)
A
mod$A ## noise added, more relistic example
```

---

Hypothesis test for IAG distribution over the ESAG distribution
*Hypothesis test for IAG distribution over the ESAG distribution*

---

## Description

The null hypothesis is whether an IAG distribution fits the data well, where the altenrative is that ESAG distribution is more suitable.

## Usage

```
iagesag(x, B = 1, tol = 1e-07)
```

### Arguments

| | |
|---|---|
| x | A numeric matrix with three columns containing the data as unit vectors in Euclidean coordinates. |
| B | The number of bootstrap re-samples. By default is set to 999. If it is equal to 1, no bootstrap is performed and the p-value is obtained throught the asymptotic distribution. |
| tol | The tolerance to accept that the Newton-Raphson algorithm used in the IAG distribution has converged. |

### Details

Essentially it is a test of rotational symmetry, whether the two $\gamma$ parameters are equal to zero. This works for spherical data only.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| statistic | The test statistic value. |
| parameter | The degrees of freedom of the test. If bootstrap was employed this is "NA". |
| p.value | The p-value of the test. |
| alternative | A character with the alternative hypothesis. |
| method | A character with the test used. |
| data.name | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

### See Also

fishkent, iagesag, pc.test, esag.mle, kent.mle,

### Examples

```
x <- rvmf(100, rnorm(3), 15)
iagesag(x)
fishkent(x, B = 1)
```

Hypothesis test for SIPC distribution over the SESPC distribution
*Hypothesis test for SIPC distribution over the SESPC distribution*

### Description

The null hypothesis is whether an SIPC distribution fits the data well, where the altenrative is that SESPC distribution is more suitable.

### Usage

```
pc.test(x, B = 1, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| `x` | A numeric matrix with three columns containing the data as unit vectors in Euclidean coordinates. |
| `B` | The number of bootstrap re-samples. By default is set to 999. If it is equal to 1, no bootstrap is performed and the p-value is obtained throught the asymptotic distribution. |
| `tol` | The tolerance to accept that the Newton-Raphson algorithm used in the IAG distribution has converged. |

### Details

Essentially it is a test of rotational symmetry, whether the two $\theta$ parameters are equal to zero. This works for spherical data only.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. If bootstrap was employed this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

## See Also

iagesag, fishkent, sespc.mle

## Examples

```
x <- rvmf(100, rnorm(3), 15)
iagesag(x)
pc.test(x)
```

---

# Hypothesis test for von Mises-Fisher distribution over Kent distribution

*Hypothesis test for von Mises-Fisher distribution over Kent distribution*

---

## Description

The null hypothesis is whether a von Mises-Fisher distribution fits the data well, where the altenrative is that Kent distribution is more suitable.

## Usage

```
fishkent(x, B = 999)
```

## Arguments

| | |
|---|---|
| x | A numeric matrix containing the data as unit vectors in Euclidean coordinates. |
| B | The number of bootstrap re-samples. By default is set to 999. If it is equal to 1, no bootstrap is performed and the p-value is obtained throught the asymptotic distribution. |

## Details

Essentially it is a test of rotational symmetry, whether Kent's ovalness parameter (beta) is equal to zero. This works for spherical data only.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. If bootstrap was employed this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

Rivest L. P. (1986). Modified Kent's statistics for testing goodness of fit for the Fisher distribution in small concentrated samples. Statistics & Probability Letters, 4(1): 1–4.

### See Also

`iagesag`, `pc.test`, `vmf.mle`, `kent.mle`

### Examples

```
x <- rvmf(100, rnorm(3), 15)
fishkent(x)
fishkent(x, B = 1)
iagesag(x)
```

---

## `Interactive 3D plot of spherical data` *Interactive 3D plot of spherical data*

---

### Description

Interactive 3D plot of spherical data.

### Usage

```
sphereplot(dat, col = NULL, bgcol = "snow")
```

### Arguments

| | |
|---|---|
| dat | A matrix with three columns, unit-vectors, spherical data. |
| col | If you want the points to appear with different colours put numbers here, otherwise leave it NULL. |
| bgcol | The color of the surface of the sphere. |

### Value

An interactive 3D plot of the spherical data will appear.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

lambert, vmf.contour, euclid

### Examples

```
x <- rvmf(100, rnorm(3), 5)
sphereplot(x)
```

---

## Inverse of Lambert's equal area projection

*Inverse of Lambert's equal area projection*

### Description

It takes some points from the cartesian coordinates and maps them onto the sphere. The inverse os the Lambert's equal area projection.

### Usage

```
lambert.inv(z, mu)
```

### Arguments

| | |
|---|---|
| z | A two- column matrix containing the Lambert's equal rea projected data. |
| mu | The mean direction of the data on the sphere. |

### Details

The data are first mapped on the sphere with mean direction equal to the north pole. Then, they are rotated to have the given mean direction. It is the inverse of the Lambert's equal are projection.

### Value

A matrix containing spherical data (unit vectors).

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent, John T. (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society. Series B (Methodological) 44(1):71–80.

### See Also

lambert

### Examples

```
m <- rnorm(3)
m <- m / sqrt( sum(m^2) )
x <- rvmf(20, m, 19)
mu <- vmf.mle(x)$mu
y <- lambert( euclid.inv(x) )
lambert.inv(y, mu)
euclid.inv(x)
```

---

Inverse of the Euclidean transformation
*Inverse of the Euclidean transformation*

---

### Description

It transforms the data from the Euclidan coordinates to latitude dn longitude.

### Usage

```
euclid.inv(U)
```

### Arguments

| | |
|---|---|
| U | A matrix of unit vectors, or even one single unit vector in three dimensions. |

### Details

It takes the matrix of unit vectors and back transforms it to latitude and longitude.

### Value

A two column matrix:

u The first column is the latitude and the second is the longitude, both expressed in degrees.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### See Also

euclid, Arotation, lambert

### Examples

```
x <- rvmf(10, rnorm(3), 10)
euclid.inv(x)
euclid( euclid.inv(x) )
x
```

---

## k-NN algorithm using the arc cosinus distance

*k-NN algorithm using the arc cosinus distance*

---

### Description

It classifies new observations to some known groups via the k-NN algorithm.

### Usage

```
dirknn(xnew, ina, x, k = 5, mesos = TRUE, parallel = FALSE, rann = FALSE)
```

### Arguments

| | |
|---|---|
| xnew | The new data whose membership is to be predicted, a numeric matrix with unit vectors. |
| ina | A variable indicating the groups of the data x. |
| x | The data, a numeric matrix with unit vectors. |
| k | The number of nearest neighbours, set to 5 by default. It can also be a vector with many values. |
| mesos | A boolean variable used only in the case of the non standard algorithm (type="NS"). Should the average of the distances be calculated (TRUE) or not (FALSE)? If it is FALSE, the harmonic mean is calculated. |

`parallel` If you want the standard -NN algorithm to take place in parallel set this equal to TRUE.

`rann` If you have large scale datasets and want a faster k-NN search, you can use kd-trees implemented in the R package "RANN". In this case you must set this argument equal to TRUE.

## Details

The standard algorithm is to keep the k nearest observations and see the groups of these observations. The new observation is allocated to the most frequent seen group. The non standard algorithm is to calculate the classical mean or the harmonic mean of the k nearest observations for each group. The new observation is allocated to the group with the smallest mean distance.

## Value

A vector including:

`g` A matrix with the predicted group(s). It has as many columns as the values of k.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alenazi A. (2019). Comparison of discriminant analysis methods on the sphere. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(4), 467–491.

## See Also

dirknn.tune, dirda, vm.nb

## Examples

```
k <- runif(4, 4, 20)
prob <- c(0.2, 0.4, 0.3, 0.1)
mu <- matrix(rnorm(16), ncol = 4)
mu <- mu / sqrt( rowSums(mu^2) )
da <- rmixvmf(200, prob, mu, k)
nu <- sample(1:200, 180)
x <- da$x[nu, ]
ina <- da$id[nu]
xx <- da$x[-nu, ]
id <- da$id[-nu]
a1 <- dirknn(xx, ina, x, k = 5, mesos = TRUE)
a2 <- dirknn(xx, ina, x, k = 5, mesos = FALSE)
table(id, a1)
table(id, a2)
```

`k-NN regression` *k-NN regression with Euclidean or (hyper-)spherical response and or predictor variables*

## Description

k-NN regression with Euclidean or (hyper-)spherical response and or predictor variables.

## Usage

```
knn.reg(xnew, y, x, k = 5, res = "eucl", estim = "arithmetic")
```

## Arguments

| | |
|---|---|
| `xnew` | The new data, new predictor variables values. A matrix with either euclidean (univariate or multivariate) or (hyper-)spherical data. If you have a circular response, say u, transform it to a unit vector via (cos(u), sin(u)). If xnew = x, you will get the fitted values. |
| `y` | The currently available data, the response variables values. A matrix with either euclidean (univariate or multivariate) or (hyper-)spherical data. If you have a circular response, say u, transform it to a unit vector via (cos(u), sin(u)). |
| `x` | The currently available data, the predictor variables values. A matrix with either euclidean (univariate or multivariate) or (hyper-)spherical data. If you have a circular response, say u, transform it to a unit vector via (cos(u), sin(u)). |
| `k` | The number of nearest neighbours, set to 5 by default. This can also be a vector with many values. |
| `res` | The type of the response variable. If it is Euclidean, set this argument equal to "res". If it is a unit vector set it to res="spher". |
| `estim` | Once the k observations whith the smallest distance are discovered, what should the prediction be? The arithmetic average of the corresponding y values be used estim="arithmetic" or their harmonic average estim="harmonic". |

## Details

This function covers a broad range of data, Euclidean and spherical, along with their combinations.

## Value

A list with as many elements as the number of values of k. Each element in the list contains a matrix (or a vector in the case of Euclidean data) with the predicted response values.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

knnreg.tune, spher.reg, spml.reg

### Examples

```
y <- iris[, 1]
x <- as.matrix(iris[, 2:4])
x <- x/ sqrt( rowSums(x^2) )  ## Euclidean response
a <- knn.reg(x, y, x, k = 5, res = "eucl", estim = "arithmetic")

y <- iris[, 2:4]
y <- y / sqrt( rowSums(y^2) ) ## Spherical response
x <- iris[, 1]
a <- knn.reg(x, y, x, k = 5, res = "spher", estim = "arithmetic")
```

---

`Lambert's equal area projection`

## *Lambert's equal area projection*

---

### Description

It calculates the Lambert's equal area projection.

### Usage

```
lambert(y)
```

### Arguments

| | |
|---|---|
| y | A two column matrix with the data. The first column is the altitude and the second is the longitude. |

### Details

The spherical data are first rotated so that their mean direction is the north pole and then are projectedt on the plane tagent to the sphere at the north pole.

### Value

A two-column matrix with the projected points.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent, John T. (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society. Series B (Methodological) 44(1):71-80.

### See Also

euclid, lambert.inv

### Examples

```
x <- rvmf(100, rnorm(3), 20)
x <- euclid.inv(x)
a <- lambert(x)
plot(a)
```

---

## Logarithm of the Kent distribution normalizing constant

*Logarithm of the Kent distribution normalizing constant*

---

### Description

Logarithm of the Kent distribution normalizing constant.

### Usage

```
kent.logcon(k, b, j = 100)
```

### Arguments

| | |
|---|---|
| k | The conencration parameter, $\kappa$. |
| b | The ovalness parameter, $\beta$. |
| j | The number of the terms in the sum to use. By default this is 100. |

### Details

It calculates logarithm of the normalising constant of the Kent distribution.

### Value

The value of the logarithm of the normalising constant of the Kent distribution.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent John (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society, Series B, 44(1): 71–80.

### See Also

`fb.saddle`, `kent.mle`

### Examples

```
kent.logcon(10, 2)
fb.saddle( c(0, 10, 0), c(0, -2, 2) )
```

---

## Many simple circular or angular regressions

*Many simple circular or angular regressions*

---

### Description

Many regressions with one circular dependent variable and one Euclidean independent variable.

### Usage

```
spml.regs(y, x, tol = 1e-07, logged = FALSE, maxiters = 100, parallel = FALSE)
```

### Arguments

| | |
|---|---|
| `y` | The dependent variable, it can be a numerical vector with data expressed in radians or it can be a matrix with two columns, the cosinus and the sinus of the circular data. The benefit of the matrix is that if the function is to be called multiple times with the same response, there is no need to transform the vector every time into a matrix. |
| `x` | A matrix with independent variable. |
| `tol` | The tolerance value to terminatate the Newton-Raphson algorithm. |
| `logged` | Do you want the logarithm of the p-value be returned? TRUE or FALSE. |
| `maxiters` | The maximum number of iterations to implement. |
| `parallel` | Do you want the calculations to take plac ein parallel? The default value if FALSE. |

### Details

The Newton-Raphson algorithm is fitted in these regression as described in Presnell et al. (1998). For each colum of x a circual regression model is fitted and the hypothesis testing of no association between y and this variable is performed.

### Value

A matrix with two columns, the test statistics and their associated (log) p-values.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Presnell B., Morrison S. P. and Littell R. C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

### See Also

spml.reg, spml.mle, iag.mle, score.cipc

### Examples

```
x <- rnorm(200)
z <- cbind(3 + 2 * x, 1 -3 * x)
y <- cbind( rnorm(100,z[, 1], 1), rnorm(100, z[, 2], 1) )
y <- y / sqrt( rowSums(y^2) )
x <- matrix( rnorm(100 * 50), ncol = 50 )
a <- Directional::spml.regs(y, x)
x <- NULL
```

---

## Maps of the world and the continents

*maps of the world and the continents*

---

### Description

It produces maps of the world and the continents.

### Usage

```
asia(title = "Asia", coords = NULL)
africa(title = "Africa", coords = NULL)
europe(title = "Europe", coords = NULL)
north.america(title = "North America", coords = NULL)
oceania(title = "Oceania", coords = NULL)
south.america(title = "South America", coords = NULL)
worldmap(title = "World map", coords = NULL)
```

### Arguments

| | |
|---|---|
| `title` | A character vector with the title of the map. |
| `coords` | If you want specific points to appear on the plot give the coordinates as a matrix, where the first column contains the longitude and the second column contains the latitude, in degrees. |

### Details

Maps of the world or the continents are produced. This are experimental functions and plot the countries with specific colouring at the moment. More functionalities will be added in the future.

### Value

A map of the selected continent or the whole world.

### Author(s)

Christos Adam.

R implementation and documentation: Christos Adam <`pada4m4@gmail.com`> and Michail Tsagris.

### See Also

`sphereplot`

### Examples

```
x <- euclid.inv( rvmf(10, rnorm(3), 5) )
```

---

## `Mixtures of rotationally symmetric distributions`
*Mixtures of rotationally symmetric distributions*

---

### Description

It performs model based clustering for circualr, spherical and hyper-spherical data assuming rotationally symetric distributions.

### Usage

```
mixvmf.mle(x, g, n.start = 5, tol = 1e-6, maxiters = 100)
mixspcauchy.mle(x, g, n.start = 5, tol = 1e-6, maxiters = 100)
mixpkbd.mle(x, g, n.start = 5, tol = 1e-6, maxiters = 100)
```

### Arguments

| | |
|---|---|
| x | A matrix with the data expressed as unit vectors. |
| g | The number of groups to fit. It must be greater than or equal to 2. |
| n.start | The number of random starts to try. See also R's built-in function kmeans for more information about this. |
| tol | The tolerance value to terminate the EM algorithm. |
| maxiters | The maximum number of iterations to perform. |

### Details

The initial step of the algorithm is not based on a spherical k-means, but on simple k-means. The results are comparable to the package movMF for the mixtures of von Mises-Fisher distributions. The other cases are mixtures of spherical Cauchy distributions or mixtures of Poisson kernel-based distributions.

### Value

A list including:

| | |
|---|---|
| param | A matrix with the mean direction, the concentration parameters and mixing probability of each group. |
| loglik | The value of the maximised log-likelihood. |
| pred | The predicted group of each observation. |
| w | The estimated probabilities of each observation to belong to each cluster. |
| iter | The number of iteration required by the EM algorithm. |
| runtime | The run time of the algorithm. A numeric vector. The first element is the user time, the second element is the system time and the third element is the elapsed time. |

### Author(s)

Michail Tsagris and Panagiotis Papastamoulis.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Panagiotis Papastamoulis <papastamoulis@aueb.gr>.

### References

Kurt Hornik and Bettina Grun (2014). movMF: An R Package for Fitting Mixtures of von Mises-Fisher Distributions http://cran.r-project.org/web/packages/movMF/vignettes/movMF.pdf

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

### See Also

rmixvmf, bic.mixvmf, mixvmf.contour

## Examples

```
k <- runif(4, 4, 6)
prob <- c(0.2, 0.4, 0.3, 0.1)
mu <- matrix(rnorm(16), ncol = 4)
mu <- mu / sqrt( rowSums(mu^2) )
x <- rmixvmf(200, prob, mu, k)$x
mixvmf.mle(x, 3)
mixvmf.mle(x, 4)
mixvmf.mle(x, 5)
```

---

`MLE of (hyper-)spherical rotationally symmetric distributions`
*MLE of (hyper-)spherical rotationally symmetric distributions*

---

## Description

MLE of (hyper-)spherical rotationally symmetric distributions.

## Usage

```
vmf.mle(x, fast = FALSE, tol = 1e-07)
multivmf.mle(x, ina, tol = 1e-07, ell = FALSE)
iag.mle(x, tol = 1e-06)
sipc.mle(x, tol = 1e-6)
acg.mle(x, tol = 1e-07)
spcauchy.mle(x, tol = 1e-06)
spcauchy.mle2(x, tol = 1e-06)
pkbd.mle(x, tol = 1e-6)
pkbd.mle2(x, tol = 1e-6)
```

## Arguments

| | |
|---|---|
| `x` | A matrix with directional data, i.e. unit vectors. |
| `fast` | IF you want a faster version, but with fewer information returned, set this equal to TRUE. |
| `ina` | A numerical vector with discrete numbers starting from 1, i.e. 1, 2, 3, 4,... or a factor variable. Each number denotes a sample or group. If you supply a continuous valued vector the function will obviously provide wrong results. |
| `ell` | This is for the multivmf.mle only. Do you want the log-likelihood returned? The default value is TRUE. |
| `tol` | The tolerance value at which to terminate the iterations. |

## Details

The vmf.mle() estimates the mean direction and concentration of a fitted von Mises-Fisher distribution.

The von Mises-Fisher distribution for groups of data is also implemented.

The acg.mle() fits the angular central Gaussian distribution. There is a constraint on the estimated covariance matrix; its trace is equal to the number of variables. An iterative algorithm takes place and convergence is guaranteed.

The iag.mle() implements MLE of the spherical projected normal distribution, for spherical and hyper-spherical data.

The spcauchy.mle() is faster than the spcacuhy.mle2() because it employs the Newton-Raphson algortihm. Both functions estimate the parameters of the spherical Cauchy distribution, for any dimension. Despite the name sounds confusing, it is implemented for arbitrary dimensions, not only the sphere. The function employs a combination of the fixed points iteration algorithm and the Brent algorithm.

The pkbd.mle() estimates the parameters of the Poisson kernel-based distribution (PKBD), for any dimension and it is faster than pkbd.mle2() for the same reason with the spcauchy.mle().

The sipc.mle() implements MLE of the spherical independent projected Cauchy distribution, for spherical data only.

## Value

For the von Mises-Fisher a list including:

`loglik` The maximum log-likelihood value.

`mu` The mean direction.

`kappa` The concentration parameter.

For the multi von Mises-Fisher a list including:

`loglik` A vector with the maximum log-likelihood values if ell is set to TRUE. Otherwise NULL is returned.

`mi` A matrix with the group mean directions.

`ki` A vector with the group concentration parameters.

For the angular central Gaussian a list including:

`iter` The number if iterations required by the algorithm to converge to the solution.

`cova` The estimated covariance matrix.

For the spherical projected normal a list including:

`iters` The number of iteration required by the Newton-Raphson.

`mesi` A matrix with two rows. The first row is the mean direction and the second is the mean vector. The first comes from the second by normalising to have unit length.

`param` A vector with the elements, the norm of mean vector, the log-likelihood and the log-likelihood of the spherical uniform distribution. The third value helps in case you want to do a log-likelihood ratio test for uniformity.

For the spherical Cauchy and the PKBD a list including:

`mesos` The mean in $R^{d+1}$. See Tsagris and Alenazy (2023) for a re-parametrization that applies in the spherical Cauchy also.

`mu` The mean direction.

`gamma` The norm of the mean in $R^{d+1}$. See Tsagris and Alenazy (2023) for a re-parametrization that applies in the spherical Cauchy also.

`rho` The concetration parameter, this takes values in [0, 1).

`loglik` The log-likelihood value.

For the SIPC a list including:

`mu` The mean direction.

`loglik` The log-likelihood value.

For the angular central Gaussian a list including:

`iter` The number of iterations performed.

`cova` The covariance matrix.

## Author(s)

Michail Tsagris and Zehao Yu.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Zehao Yu <zehaoy@email.sc.edu>.

## References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Sra S. (2012). A short note on parameter approximation for von Mises-Fisher distributions: and a fast implementation of $I_s(x)$. Computational Statistics, 27(1): 177–190.

Tyler D. E. (1987). Statistical analysis for the angular central Gaussian distribution on the sphere. Biometrika 74(3): 579–589.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28: 689–697.

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248. https://arxiv.org/pdf/1510.07679.pdf

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Sablica L., Hornik K. and Leydold J. (2023). Efficient sampling from the PKBD distribution. Electronic Journal of Statistics, 17(2): 2180–2209.

Zehao Yu and Xianzheng Huang (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statististics, 18(1): 301–334.

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

### See Also

racg, rvmf

### Examples

```
m <- c(0, 0, 0, 0)
s <- cov(iris[, 1:4])
x <- racg(100, s)
mod <- acg.mle(x)
mod
cov2cor(mod$cova)  ## estimated covariance matrix turned into a correlation matrix
cov2cor(s)  ## true covariance matrix turned into a correlation matrix
vmf.mle(x)
x <- rbind( rvmf(100,rnorm(4), 10), rvmf(100,rnorm(4), 20) )
a <- multivmf.mle(x, rep(1:2, each = 100) )
```

---

## MLE of some circular distributions

*MLE of some circular distributions*

---

### Description

MLE of some circular distributions.

### Usage

```
spml.mle(x, rads = FALSE, tol = 1e-07)
wrapcauchy.mle(x, rads = FALSE, tol = 1e-07)
wrapnormal.mle(x, rads = FALSE)
circexp.mle(x, rads = FALSE, tol = 1e-06)
circbeta.mle(x, rads = FALSE)
cardio.mle(x, rads = FALSE)
ggvm.mle(phi, rads = FALSE)
cipc.mle(x, rads = FALSE, tol = 1e-6)
gcpc.mle(x, rads = FALSE)
gcpc.mle2(x, rads = FALSE)
mmvm.mle(x, N, rads = FALSE)
wraplaplace.mle(x, rads = FALSE)
```

### Arguments

| | |
|---|---|
| `x` | A numerical vector with the circular data. They can either be expressed in radians or in degrees. |
| `phi` | A numerical vector with the circular data. They can either be expressed in radians or in degrees. |
| `N` | The number of modes to consider in the multi-modal von Mises distribution. |
| `rads` | If the data are in radians set this to TRUE. |
| `tol` | The tolerance level to stop the iterative process of finding the MLEs. |

### Details

The parameters of the bivariate angular Gaussian (spml.mle), wrapped Cauchy, circular exponential, cardioid, circular beta, geometrically generalised von Mises, CIPC (reparametrised version of the wrapped Cauchy), GCPC (generalisation of the CIPC), multi-modal von Mises and wrapped Laplace distributions are estimated. For the Wrapped Cauchy, the iterative procedure described by Kent and Tyler (1988) is used. The Newton-Raphson algortihm for the angular Gaussian is described in the regression setting in Presnell et al. (1998). The circular exponential is also known as wrapped exponential distribution. The gcpc.mle() and gcpc.mle2() are both for the GCPC distribution. The first uses the Euclidean coordinates, whereas the second uses the polar coordinates, of the density. In cases of bimodality, the first function could be trapped in a local maximum, whereas the second function will not be trapped (see Alzeley and Tsagris, 2026).

### Value

A list including:

| | |
|---|---|
| `iters` | The iterations required until convergence. |
| `loglik` | The value of the maximised log-likelihood. |
| `param` | A vector consisting of the estimates of the two parameters, the mean direction for both distributions and the concentration parameter $\kappa$ and the $\rho$ for the von Mises (and the multi-modal von Mises) and the wrapped Cauchy and normal respectively. For the circular beta this contains the mean angle and the $\alpha$ and $\beta$ parameters. For the cardioid distribution this contains the $\mu$ and $\rho$ parameters. For the generalised von Mises this is a vector consisting of the $\zeta$, $\kappa$, $\mu$ and $\alpha$ parameters of the generalised von Mises distribution as described in Equation (2.7) of Dietrich and Richter (2017). |
| `gamma` | The norm of the mean vector of the angular Gaussian, the CIPC and the GCPC distributions. |
| `mu` | The mean vector of the angular Gaussian, the CIPC and the GCPC distributions. |
| `mumu` | In the case of "angular Gaussian distribution this is the mean angle in radians. |
| `circmu` | In the case of the CIPC and the GCPC this is the mean angle in radians. |
| `rho` | For the GCPC distribution this is the eigenvalue of the covariance matrix, or the covariance determinant. |
| `lambda` | The lambda parameter of the circular exponential distribution. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Sra S. (2012). A short note on parameter approximation for von Mises-Fisher distributions: and a fast implementation of $I_s(x)$. Computational Statistics, 27(1): 177–190.

Presnell Brett, Morrison Scott P. and Littell Ramon C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

Kent J. and Tyler D. (1988). Maximum likelihood estimation for the wrapped Cauchy distribution. Journal of Applied Statistics, 15(2): 247–254.

Dietrich T. and Richter W. D. (2017). Classes of geometrically generalized von Mises distributions. Sankhya B, 79(1): 21–59.

https://en.wikipedia.org/wiki/Wrapped_exponential_distribution

Jammalamadaka S. R. and Kozubowski T. J. (2003). A new family of circular models: The wrapped Laplace distributions. Advances and Applications in Statistics, 3(1), 77–103.

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Alzeley O. & Tsagris M. (2026). On the generalized circular projected Cauchy distribution. https://arxiv.org/pdf/2603.04030.

Barnett M. J. and Kingston R. L. (2024). A note on the Hendrickson-Lattman phase probability distribution and its equivalence to the generalized von Mises distribution. Journal of Applied Crystallography, 57(2).

Lopez-Custodio P. C. (2024). A cheat sheet for probability distributions of orientational data. arXiv:2412.08934.

### See Also

circ.summary, purka.mle, rvonmises, vmf.mle, rvmf

### Examples

```
x <- rvonmises(1000, 3, 9)
spml.mle(x, rads = TRUE)
wrapcauchy.mle(x, rads = TRUE)
circexp.mle(x, rads = TRUE)
```

`MLE of some circular distributions with multiple samples`
*MLE of some circular distributions with multiple samples*

### Description

MLE of some circular distributions with multiple samples.

### Usage

```
multivm.mle(x, ina, tol = 1e-07, ell = FALSE)
multispml.mle(x, ina, tol = 1e-07, ell = FALSE)
```

### Arguments

| | |
|---|---|
| `x` | A numerical vector with the circular data. They must be expressed in radians. For the "spml.mle" this can also be a matrix with two columns, the cosinus and the sinus of the circular data. |
| `ina` | A numerical vector with discrete numbers starting from 1, i.e. 1, 2, 3, 4,... or a factor variable. Each number denotes a sample or group. If you supply a continuous valued vector the function will obviously provide wrong results. |
| `tol` | The tolerance level to stop the iterative process of finding the MLEs. |
| `ell` | Do you want the log-likelihood returned? The default value is FALSE. |

### Details

The parameters of the von Mises and of the bivariate angular Gaussian distributions are estimated for multiple samples.

### Value

A list including:

| | |
|---|---|
| `iters` | The iterations required until convergence. This is returned in the wrapped Cauchy distribution only. |
| `loglik` | A vector with the value of the maximised log-likelihood for each sample. |
| `mi` | For the von Mises, this is a vector with the means of each sample. For the angular Gaussian (spml), a matrix with the mean vector of each sample |
| `ki` | A vector with the concentration parameter of the von Mises distribution at each sample. |
| `gi` | A vector with the norm of the mean vector of the angular Gaussian distribution at each sample. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris `<mtsagris@uoc.gr>`.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Sra S. (2012). A short note on parameter approximation for von Mises-Fisher distributions: and a fast implementation of $I_s(x)$. Computational Statistics, 27(1): 177–190.

Presnell Brett, Morrison Scott P. and Littell Ramon C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

Kent J. and Tyler D. (1988). Maximum likelihood estimation for the wrapped Cauchy distribution. Journal of Applied Statistics, 15(2): 247–254.

### See Also

`colspml.mle`, `purka.mle`

### Examples

```
y <- rcauchy(100, 3, 1)
x <- y
ina <- rep(1:2, 50)
multivm.mle(x, ina)
multispml.mle(x, ina)
```

---

`MLE of the ESAG distribution in arbitrary dimensions`

## *MLE of the ESAG distribution*

---

### Description

MLE of the ESAG distribution.

### Usage

```
esag.mle(y, full = FALSE, tol = 1e-06)
ESAGd.mle(y, full = FALSE)
```

### Arguments

| | |
|---|---|
| `y` | A matrix with the data expressed in Euclidean coordinates, i.e. unit vectors. The function esag.mle() works for spherical data, whereas ESAGd.mle() is for spherical and hyper-spherical data. |
| `full` | If you want some extra information, the inverse of the covariance matrix, the $rho$ parameter (smallest eigenvalue of the covariance matrix) and the angle of rotation $\psi$, set this equal to TRUE. Otherwise leave it FALSE. |
| `tol` | A tolerance value to stop performing successive optimizations. |

## Details

MLE of the MLE of the ESAG distributiontribution, on the sphere, is implemented. ESAG stands for Elliptically Symmetric Angular Gaussian and it was suugested by Paine et al. (2018). Unlike the projected normal distribution this is rotationally symmetric and is a competitor of the spherical Kent distribution (which is also elliptically symmetric). ESAG was then generalized to arbitrary dimensions by Yu and Huang (2024).

## Value

A list including:

| | |
|---|---|
| `mu` | The mean vector. |
| `gam` | The $\gamma$ parameters. |
| `loglik` | The log-likelihood value. |
| `vinv` | The inverse of the covariance matrix. It is returned if the argument "full" is TRUE. |
| `rho` | The $\rho$ parameter (smallest eigenvalue of the covariance matrix). It is returned if the argument "full" is TRUE in the esag.mle(). |
| `psi` | The angle of rotation $\psi$ set this equal to TRUE. It is returned if the argument "full" is TRUE in esag.mle(). |
| `lambda` | The $d-1$ eigenvalues of the covariance matrix of the ESAG distribution in arbitrary dimensions. This is returned if "full" is set to TRUE in the ESAGd.mle(). |
| `iag.loglik` | The log-likelihood value of the isotropic angular Gaussian distribution in the esag.mle(). That is, the projected normal distribution which is rotationally symmetric. |

## Author(s)

Michail Tsagris and Zehao Yu.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Zehao Yu <zehaoy@email.sc.edu>.

## References

Zehao Yu and Xianzheng Huang (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statistics, 18(1): 301–334.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

`desag`, `resag`, `iag.mle`, `kent.mle`, `acg.mle`, `circ.summary`, `sphereplot`

## Examples

```
m <- colMeans( as.matrix( iris[, 1:3] ) )
y <- resag(1000, m, c(1, 0.5) )
esag.mle(y)

m <- colMeans( as.matrix( iris[, 1:4] ) )
y <- rESAGd(1000, m, c(1, 0.5, -1, 1, -0.5) )
ESAGd.mle(y)
```

---

MLE of the Kent distribution

*MLe of the Kent distribution*

---

## Description

It estimates the concentration and the ovalness parameter of some directional data assuming the Kent distribution. The mean direction and major and minor axes are also estimated.

## Usage

```
kent.mle(x)
```

## Arguments

| | |
|---|---|
| x | A matrix containing spherical data in Euclidean coordinates. |

## Details

The Kent distribution is fitted to some data and its parameters are estimated.

## Value

A list including:

| | |
|---|---|
| runtime | The run time of the procedure. |
| G | A 3 x 3 matrix whose first column is the mean direction. The second and third columns are the major and minor axes respectively. |
| param | A vector with the concentration $\kappa$ and ovalness $\beta$ parameters and the angle $\psi$ used to rotate **H** and hence estimate **G** as in Kent (1982). |
| logcon | The logarithm of the normalising constant, using the third type approximation (Kume and Wood, 2005). |
| loglik | The value of the log-likelihood. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent John (1982). The Fisher-Bingham distribution on the sphere. Journal of the Royal Statistical Society, Series B, 44(1): 71–80.

Kume Alfred and Wood Andrew T.A. (2005). Saddlepoint approximations for the Bingham and Fisher-Bingham normalizing constants. Biometrika, 92(2):465–476

### See Also

kent.mle, fb.saddle, vmf.mle, wood.mle, sphereplot

### Examples

```
x <- rvmf(200, rnorm(3), 15)
kent.mle(x)
vmf.mle(x)
A <- diag( c(-5, 0, 5) )
x <- rfb(200, 15, rnorm(3), A)
kent.mle(x)
vmf.mle(x)
```

---

MLE of the Matrix Fisher distribution on SO(3)
*MLE of the Matrix Fisher distribution on SO(3)*

---

### Description

It returns the maximum likelihood estimate of the Matrix Fisher parameter F(3x3).

### Usage

```
matrixfisher.mle(X)
```

### Arguments

| | |
|---|---|
| X | An array containing rotation matrices in SO(3). |

### Value

The components of $svd(\bar{X})$.

### Author(s)

Anamul Sajib and Chris Fallaize.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd> and Chris Fallaize.

### References

Prentice M. J. (1986). Orientation statistics without parametric assumptions. Journal of the Royal Statistical Society. Series B: Methodological 48(2): 214–222.

### See Also

rmatrixfisher

### Examples

```
F <- 10^(-1) * matrix( c(85, 11, 41, 78, 39, 60, 43, 64, 48), ncol = 3 )  ### An arbitrary F matrix
X <- rmatrixfisher(5000, F)
matrixfisher.mle(X)
svd(F)
```

---

MLE of the Purkayashta distribution

## *MLE of the Purkayashta distribution*

---

### Description

MLE of the Purkayashta distribution.

### Usage

```
purka.mle(x, tol = 1e-07)
```

### Arguments

| | |
|---|---|
| x | A numerical vector with data expressed in radians or a matrix with spherical data. |
| tol | The tolerance value to terminate the Brent algorithm. |

### Details

MLE of the Purkayastha distribution is performed.

### Value

A list including:

| | |
|---|---|
| theta | The median direction. |
| circtheta | In case of circular data the circular mean is also returned. |
| alpha | The concentration parameter. |
| loglik | The log-likelihood. |
| alpha.sd | The standard error of the concentration parameter. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83.

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics-Theory and Methods, 19(6): 1973–1986.

### See Also

circ.cor1

### Examples

```
x <- cbind( rnorm(100,1,1), rnorm(100, 2, 1) )
x <- x / sqrt(rowSums(x^2))
purka.mle(x)
```

---

MLE of the SESPC distribution
*MLE of the SESPC distribution*

---

### Description

MLE of the SESPC distribution.

### Usage

```
sespc.mle(y, full = FALSE, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| y | A matrix with the data expressed in Euclidean coordinates, i.e. unit vectors. |
| full | If you want some extra information, the inverse of the covariance matrix, set this equal to TRUE. Otherwise leave it FALSE. |
| tol | A tolerance value to stop performing successive optimizations. |

### Details

MLE of the SESPC distribution is implemented. SESPC stands for Spherical Elliptically Symmetric Projected Cauchy and it was suugested by Tsagris and Alzeley (2025). Unlike the spherical independent projected Cauchy distribution this is rotationally symmetric and is a competitor of the spherical ESAG and Kent distributions (which are also ellitpically symmetric).

## Value

A list including:

| | |
|---|---|
| mu | The mean vector in $R^3$. |
| theta | The two $\theta$ parameters. |
| loglik | The log-likelihood value. |
| vinv | The inverse of the covariance matrix. It is returned if the argument "full" is TRUE. |
| lambda | The $\lambda_2$ parameter (smallest eigenvalue of the covariance matrix). It is returned if the argument "full" is TRUE. |
| psi | The angle of rotation $\psi$ set this equal to TRUE. It is returned if the argument "full" is TRUE. |
| sipc.loglik | The log-likelihood value of the isotropic prohected Cuchy distribution, which is rotationally symmetric. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

dsespc, rsespc, sipc.mle, esag.mle, spher.sespc.contour

## Examples

```
m <- colMeans( as.matrix( iris[,1:3] ) )
y <- rsespc(1000, m, c(1,0.5) )
sespc.mle(y)
```

MLE of the Wood bimodal distribution on the sphere
*MLE of the Wood bimodal distribution on the sphere*

## Description

It estimates the parameters of the Wood bimodal distribution.

## Usage

```
wood.mle(y)
```

## Arguments

| | |
|---|---|
| y | A matrix containing two columns. The first one is the latitude and the second is the longitude, both expressed in degrees. |

## Details

The Wood distribution is fitted to some data and its parameters are estimated. It is a bimodal distribution which contains 5 parameters, just like the Kent distribution.

## Value

A list including:

| | |
|---|---|
| info | A 5 x 3 matrix containing the 5 parameters, $\gamma$, $\delta$, $\alpha$, $\beta$ and $\kappa$ along with their corresponding 95% confidence intervals all expressed in degrees. |
| modes | The two axis of the modes of the distribution expressed in degrees. |
| unitvectors | A 3 x 3 matrix with the 3 unit vectors associated with the $\gamma$ and $\delta$ parameters. |
| loglik | The value of the log-likelihood. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Wood A.T.A. (1982). A bimodal distribution on the sphere. Journal of the Royal Statistical Society, Series C, 31(1): 52–58.

## See Also

kent.mle, esag.mle, vmf.mle, sphereplot

### Examples

```
x <- rvmf(100, rnorm(3), 15)
x <- euclid.inv(x)
wood.mle(x)
```

---

## Naive Bayes classifiers for circular data

*Naive Bayes classifiers for directional data*

---

### Description

Naive Bayes classifiers for directional data.

### Usage

```
vm.nb(xnew = NULL, x, ina, tol = 1e-07)
spml.nb(xnew = NULL, x, ina, tol = 1e-07)
```

### Arguments

| | |
|---|---|
| xnew | A numerical matrix with new predictor variables whose group is to be predicted. Each column refers to an angular variable. |
| x | A numerical matrix with observed predictor variables. Each column refers to an angular variable. |
| ina | A numerical vector with strictly positive numbers, i.e. 1,2,3 indicating the groups of the dataset. Alternatively this can be a factor variable. |
| tol | The tolerance value to terminate the Newton-Raphson algorithm. |

### Details

Each column is supposed to contain angular measurements. Thus, for each column a von Mises distribution or an circular angular Gaussian distribution is fitted. The product of the densities is the joint multivariate distribution.

### Value

A list including:

| | |
|---|---|
| mu | A matrix with the mean vectors expressed in radians. |
| mu1 | A matrix with the first set of mean vectors. |
| mu2 | A matrix with the second set of mean vectors. |
| kappa | A matrix with the kappa parameters for the vonMises distribution or with the norm of the mean vectors for the circular angular Gaussian distribution. |
| ni | The sample size of each group in the dataset. |
| est | The estimated group of the xnew observations. It returns a numerical value back regardless of the target variable being numerical as well or factor. Hence, it is suggested that you do \"as.numeric(ina)\" in order to see what is the predicted class of the new data. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

vmnb.pred

### Examples

```
x <- matrix( runif( 100, 0, 1 ), ncol = 2 )
ina <- rbinom(50, 1, 0.5) + 1
a <- vm.nb(x, x, ina)
```

---

## Normalised spatial median for directional data

*Normalised spatial median for directional data*

---

### Description

Normalised spatial median for directional data.

### Usage

```
nsmedian(x, tol = 1e-07)
```

### Arguments

| | |
|---|---|
| x | A matrix with Euclidean data, continuous variables. |
| tol | A tolerance level to terminate the process. |

### Details

The spatial median, using a fixed point iterative algorithm, for Euclidean data is calculated. It is a robust location estimate. Then it is normalised to become a unit vector. Generally speaking this might be a better alternative than then mediandir.

### Value

A vector with the spatial median.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Ducharme G. R. and Milasevic P. (1987). Spatial median and directional data. Biometrika, 74(1), 212-215.

Jyrki Mottonen, Klaus Nordhausen and Hannu Oja (2010). Asymptotic theory of the spatial median. In Nonparametrics and Robustness in Modern Statistical Inference and Time Series Analysis: A Festschrift in honor of Professor Jana Jureckova.

T. Karkkaminen and S. Ayramo (2005). On computation of spatial median for robust data mining. Evolutionary and Deterministic Methods for Design, Optimization and Control with Applications to Industrial and Societal Problems, EUROGEN 2005, R. Schilling, W.Haase, J. Periaux, H. Baier, G. Bugeda (Eds) FLM, Munich. http://users.jyu.fi/~samiayr/pdf/ayramo_eurogen05.pdf

### See Also

mediandir

### Examples

```
m <- rnorm(3)
m <- m / sqrt( sum(m^2) )
x <- rvmf(100, m, 10)
nsmedian(x)
mediandir(x)
```

---

## Permutation based 2-sample mean test for (hyper-)spherical data

*Permutation based 2-sample mean test for (hyper-)spherical data*

---

### Description

Permutation based 2-sample mean test for (hyper-)spherical data.

### Usage

```
hcf.perm(x1, x2, B = 999)
lr.perm(x1, x2, B = 999)
hclr.perm(x1, x2, B = 999)
embed.perm(x1, x2, B = 999)
het.perm(x1, x2, B = 999)
```

### Arguments

| | |
|---|---|
| x1 | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| x2 | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| B | The number of permutations to perform. |

### Details

The high concentration (hcf.perm), log-likelihood ratio (lr.perm), high concentration log-likelihood ratio (hclr.perm), embedding approach (embed.perm) or the non equal concentration parameters approach (het.perm) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are permutation based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1), 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

`hcf.boot`, `hcf.aov`, `spherconc.test`, `conc.test`

### Examples

```
x <- rvmf(60, rnorm(3), 15)
ina <- rep(1:2, each = 30)
x1 <- x[ina == 1, ]
x2 <- x[ina == 2, ]
hcf.perm(x1, x2)
lr.perm(x1, x2)
het.boot(x1, x2)
```

`Permutation based 2-sample mean test for circular data`
*Permutation based 2-sample mean test for circular data*

## Description

Permutation based 2-sample mean test for circular data.

## Usage

```
hcfcirc.perm(u1, u2, rads = TRUE, B = 999)
hetcirc.perm(u1, u2, rads = TRUE, B = 999)
lrcirc.perm(u1, u2, rads = TRUE, B = 999)
hclrcirc.perm(u1, u2, rads = TRUE, B = 999)
embedcirc.perm(u1, u2, rads = TRUE, B = 999)
```

## Arguments

| | |
|---|---|
| `u1` | A numeric vector containing the data of the first sample. |
| `u2` | A numeric vector containing the data of the first sample. |
| `rads` | If the data are in radians, this should be TRUE and FALSE otherwise. |
| `B` | The number of permutations to perform. |

## Details

The high concentration (hcfcirc.perm), log-likelihood ratio (lrcirc.perm), high concentration log-likelihood ratio (hclrcirc.perm), embedding approach (embedcirc.perm) or the non equal concentration parameters approach (hetcirc.perm) is used.

## Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. Since these are permutation based tests this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Tsagris M. and Alenazi A. (2024). An investigation of hypothesis testing procedures for circular and spherical mean vectors. Communications in Statistics-Simulation and Computation, 53(3): 1387–1408.

### See Also

`hcf.circaov`, `het.aov`

### Examples

```
u1 <- rvonmises(20, 2.4, 5)
u2 <- rvonmises(20, 2.4, 10)
hcfcirc.perm(u1, u2)
lrcirc.perm(u1, u2)
```

---

## `Prediction in discriminant analysis based on some distributions`

*Prediction of a new observation using discriminant analysis based on some distributions*

---

### Description

Prediction of a new observation using discriminant analysis based on some distributions.

### Usage

```
dirda(xnew, x, ina, type = c("vmf", "iag", "esag", "kent", "sc", "pkbd", "purka") )
```

### Arguments

| | |
|---|---|
| `xnew` | The new observation(s) (unit vector(s)) whose group is to be predicted. |
| `x` | A data matrix with unit vectors, i.e. spherical directional data. |
| `ina` | A vector indicating the groups of the data y. |
| `type` | The type of classifier to use. The avaliable options are "vmf" (von Mises-Fisher distribution), "iag" (IAG distribution), "esag" (ESAG distribution), "kent" (Kent distribution), "sc" and "sc2" (spherical Cauchy distribution), "pkbd" and "pkbd2" (Poisson kernel-based distribution), and "purka" (Purkayastha distribution). The difference between "sc" and "sc2" and between "pkbd" and "pkbd2" is that the first uses the Newton-Raphson algorithm and it is faster, whereas the second uses a hybrid algorithm that does not require the Hessian matrix, but in large dimensions the second will be faster. You can chose any of them or all of them. Note that "kent" works only with spherical data. |

## Details

Prediction of the class of a new (hyper-)spherical vector assuming some distributions.

## Value

A vector with the predicted group of each new observation.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alenazi A. (2019). Comparison of discriminant analysis methods on the sphere. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(4): 467–491.

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

Morris J. E. and Laycock P. J. (1974). Discriminant analysis of directional data. Biometrika, 61(2): 335–341.

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

dirda.cv, vm.nb, dirknn, knn.reg

## Examples

```
m1 <- rnorm(3)
m2 <- rnorm(3) + 0.5
x <- rbind( rvmf(100, m1, 3), rvmf(80, m2, 5) )
ina <- c( rep(1,100), rep(2, 80) )
xnew <- rbind(rvmf(10, m1, 10), rvmf(10, m2, 5))
id <- rep(1:2, each = 10)
g <- dirda(xnew, x, ina, type = "vmf")
table(id, g[, 1])
```

---

Prediction with some naive Bayes classifiers for circular data

*Prediction with some naive Bayes classifiers for circular data*

---

## Description

Prediction with some naive Bayes classifiers for circular data.

### Usage

```
vmnb.pred(xnew, mu, kappa, ni)
spmlnb.pred(xnew, mu1, mu2, ni)
```

### Arguments

| | |
|---|---|
| xnew | A numerical matrix with new predictor variables whose group is to be predicted. Each column refers to an angular variable. |
| mu | A matrix with the mean vectors expressed in radians. |
| mu1 | A matrix with the first set of mean vectors. |
| mu2 | A matrix with the second set of mean vectors. |
| kappa | A matrix with the kappa parameters for the vonMises distribution or with the norm of the mean vectors for the circular angular Gaussian distribution. |
| ni | The sample size of each group in the dataset. |

### Details

Each column is supposed to contain angular measurements. Thus, for each column a von Mises distribution or an circular angular Gaussian distribution is fitted. The product of the densities is the joint multivariate distribution.

### Value

A numerical vector with 1, 2, ... denoting the predicted group.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

vm.nb

### Examples

```
x <- matrix( runif( 100, 0, 1 ), ncol = 2 )
ina <- rbinom(50, 1, 0.5) + 1
a <- vm.nb(x, x, ina)
a2 <- vmnb.pred(x, a$mu, a$kappa, a$ni)
```

Projections based test of uniformity
*Projections based test of uniformity*

## Description

It checkes whether the data are uniformly distributed on the circle or the (hyper-)sphere.

## Usage

```
ptest(x, B = 100)
```

## Arguments

| | |
|---|---|
| x | A matrix containing the data, unit vectors. |
| B | The number of random uniform projections to use. |

## Details

For more details see Cuesta-Albertos, Cuevas and Fraiman (2009).

## Value

A list including:

| | |
|---|---|
| pvalues | The p-values of the Kolmogorov-Smirnov tests. |
| pvalue | The p-value of the test based on the Benjamini and Heller (2008) procedure. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Cuesta-Albertos J. A., Cuevas A. and Fraiman, R. (2009). On projection-based tests for directional and compositional data. Statistics and Computing, 19: 367–380.

Benjamini Y. and Heller R. (2008). Screening for partial conjunction hypotheses. Biometrics, 64(4): 1215–1222.

## See Also

rayleigh, kuiper

## Examples

```
x <- rvmf(100, rnorm(5), 1)  ## Fisher distribution with low concentration
ptest(x)
```

Random sample of matrices in SO(p)

*Random sample of matrices in SO(p)*

## Description

Random sample of matrices in SO(p).

## Usage

```
rsop(n, p)
```

## Arguments

| | |
|---|---|
| n | The sample size, the number of matrices you want to generate. |
| p | The dimensionality of the matrices. |

## Details

The idea is very simple. Start with a unit vector pointing at the north pole (1,0,...,0). Then generate random numbers from a standard normal and scale them so that they have a unit length. To put it differently, a sample of n values from the uniform distribution on the sphere is generated. Then calculate the rotation matrix required to go from the north pole to each of a generated vector.

## Value

If n = 1 one matrix is returned. If n is greater than 1, an array with n matrices inside.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Amaral G.J.A., Dryden I.L. and Wood A.T.A. (2007). Pivotal Bootstrap Methods for k-Sample Problems in Directional Statistics and Shape Analysis. Journal of the American Statistical Association, 102(478): 695–707.

## See Also

rotation, Arotation, rot.matrix

## Examples

```
x1 <- rsop(1, 3)
x2 <- rsop(10, 3)
x3 <- rsop(100, 10)
```

---

Rayleigh's test of uniformity

*Rayleigh's test of uniformity*

---

## Description

It checkes whether the data are uniformly distributed on the circle or the (hyper-)sphere.

## Usage

```
rayleigh(x, modif = TRUE, B = 999)
```

## Arguments

| | |
|---|---|
| x | A matrix containing the data, unit vectors. |
| modif | If modif is TRUE, the modification as suggested by Jupp (2001) is used. |
| B | If B is greater than 1, bootstap calibation os performed. If it is equal to 1, classical theory is used. |

## Details

The Rayleigh test of uniformity is not the best, when there are two antipodal mean directions. In this case it will fail. It is good to test whether there is one mean direction or not. To put it differently, it tests whether the concentration parameter of the Fisher distribution is zero or not.

## Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| statistic | The test statistic value. |
| parameter | The degrees of freedom of the test. If bootstrap was employed this is "NA". |
| p.value | The p-value of the test. |
| alternative | A character with the alternative hypothesis. |
| method | A character with the test used. |
| data.name | A character vector with two elements. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Jupp, P. E. (2001). Modifications of the Rayleigh and Bingham tests for uniformity of directions. Journal of Multivariate Analysis, 77(2): 1-20.

Rayleigh, L. (1919). On the problem of random vibrations, and of random flights in one, two, or three dimensions. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 37(220): 321–347.

### See Also

`ptest`, `kuiper`, `iagesag`

### Examples

```
x <- rvmf(100, rnorm(5), 1)  ## Fisher distribution with low concentration
rayleigh(x)
```

---

`Read a file as a Filebacked Big Matrix`

## *Read a file as a Filebacked Big Matrix*

---

### Description

Read a file as a Filebacked Big Matrix.

### Usage

```
read.fbm(file, select)
```

### Arguments

| | |
|---|---|
| `file` | The File to read. |
| `select` | Indices of columns to read (sorted). The length of select will be the number of columns of the resulting FBM. |

### Details

The functions read a file as a Filebacked Big Matrix object. For more information see the "bigstatsr" package.

### Value

A Filebacked Big Matrix object.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

vmf.mle, kent.mle

### Examples

```
x <- matrix( runif(50 * 20, 0, 2*pi), ncol = 20 )
```

---

Rotation axis and angle of rotation given a rotation matrix
*Rotation axis and angle of rotation given a rotation matrix*

---

### Description

Given a 3 x 3 rotation matrix, the angle and the axis of rotation are calculated.

### Usage

```
Arotation(A)
```

### Arguments

| | |
|---|---|
| A | A 3 x 3 rotation matrix. |

### Details

If the user does not supply a rotation matrix a message will appear.

### Value

A list including:

| | |
|---|---|
| angle | The angle of rotation expressed in degrees. |
| axis | The axis of rotation. A vector of two components, latitude and longitude, expressed in degrees. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Course webpage of Howard E. Haber. http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf

Ted Chang (1986). Spherical Regression. Annals of Statistics, 14(3): 907–924.

## See Also

rot.matrix, rotation, rsop

## Examples

```
ksi <- c(25.31, 24.29)
theta <- 2.38
A <- rot.matrix(ksi, theta, rads = FALSE)
A
Arotation(A)
```

---

# Rotation matrix from a rotation axis and angle of rotation

*Rotation matrix from a rotation axis and angle of rotation*

---

## Description

It calculates a rotation matrix from a rotation axis and angle of rotation.

## Usage

```
rot.matrix(ksi, theta, rads = FALSE)
```

## Arguments

| | |
|---|---|
| ksi | The rotation axis, a vector with two elements, the first is the latitude and the second is the longitude. |
| theta | The angle of rotation. |
| rads | If both the ksi and theta are in rads, this should be TRUE. If both the ksi and theta are in degrees, this should be FALSE. |

## Details

The function accepts as arguments the rotation axis and the angle of rotation and it calcualtes the requested rotation matrix.

## Value

A 3 x 3 rotation matrix.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Course webpage of Howard E. Haber. http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf

Ted Chang (1986). Spherical Regression. Annals of Statistics, 14(3): 907–924.

## See Also

Arotation, rotation, rsop

## Examples

```
ksi <- c(25.31, 24.29)
theta <- 2.38
A <- rot.matrix(ksi, theta, rads = FALSE)
A
Arotation(A)
```

---

Rotation matrix on SO(3) from three Euler angles
*Construct a rotation matrix on SO(3) from the Euler angles.*

---

## Description

It forms a rotation matrix X on SO(3) by using three Euler angles $(\theta_{12}, \theta_{13}, \theta_{23})$, where X is defined as $X = R_z(\theta_{12}) \times R_y(\theta_{13}) \times R_x(\theta_{23})$. Here $R_x(\theta_{23})$ means a rotation of $\theta_{23}$ radians about the x axis.

## Usage

```
eul2rot(theta.12, theta.23, theta.13)
```

## Arguments

| | |
|---|---|
| theta.12 | An Euler angle, a number which must lie in $(-\pi, \pi)$. |
| theta.23 | An Euler angle, a number which must lie in $(-\pi, \pi)$. |
| theta.13 | An Euler angle, a number which must lie in $(-\pi/2, \pi/2)$. |

## Details

Given three euler angles a rotation matrix X on SO(3) is formed using the transformation according to Green and Mardia (2006) which is defined above.

### Value

A roation matrix.

### Author(s)

Anamul Sajib <sajibstat@du.ac.bd>.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd>.

### References

Green, P. J. and Mardia, K. V. (2006). Bayesian alignment using hierarchical models, with applications in proteins bioinformatics. Biometrika, 93(2):235–254.

### See Also

rot2eul

### Examples

```
# three euler angles

theta.12 <- sample( seq(-3, 3, 0.3), 1 )
theta.23 <- sample( seq(-3, 3, 0.3), 1 )
theta.13 <- sample( seq(-1.4, 1.4, 0.3), 1 )

theta.12 ; theta.23 ; theta.13

X <- eul2rot(theta.12, theta.23, theta.13)
X  # A rotation matrix
det(X)

e <- rot2eul(X)$v1

theta.12 <- e[3]
theta.23 <- e[2]
theta.13 <- e[1]

theta.12 ; theta.23 ; theta.13
```

---

## Rotation matrix to rotate a spherical vector along the direction of another

*Rotation matrix to rotate a spherical vector along the direction of another*

---

### Description

A rotation matrix is calculated to rotate a unit vector along the direction of another.

### Usage

```
rotation(a, b)
```

### Arguments

| | |
|---|---|
| a | The initial unit vector. |
| b | The target unit vector. |

### Details

The function calcualtes a rotation matrix given two vectors. This rotation matrix is the connection between the two spherical only, vectors.

### Value

A rotation matrix whose dimension is equal to the length of the unit vectors.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Amaral G.J.A., Dryden I.L. and Wood A.T.A. (2007). Pivotal Bootstrap Methods for k-Sample Problems in Directional Statistics and Shape Analysis. Journal of the American Statistical Association, 102(478): 695–707.

### See Also

Arotation, rot.matrix, lambert, lambert.inv, rsop

### Examples

```
a <- rnorm(3)
a <- a/sqrt(sum(a^2))
b <- rnorm(3)
b <- b/sqrt(sum(b^2))
A <- rotation(a, b)
A
a  ;  b
a %*% t(A)

a <- rnorm(7)
a <- a/sqrt(sum(a^2))
b <- rnorm(7)
b <- b/sqrt(sum(b^2))
A <- rotation(a, b)
A
```

```
a  ;  b
a %*% t(A)
```

---

Saddlepoint approximations of the Fisher-Bingham distributions
*Saddlepoint approximations of the Fisher-Bingham distributions*

---

### Description

It calculates the logarithm of the normalising constant of the Fisher-Bingham distribution.

### Usage

```
fb.saddle(gam, lam)
```

### Arguments

| | |
|---|---|
| gam | A numeric vector containing the parameters of the Fisher part. |
| lam | All the eigenvalues of the Bingham part. Not just the non zero ones. |

### Details

It calculate the three approximations given by Kume and Wood (2005) and it uses the Fisher-Bingham parametrization of that paper.

### Value

A list including:

| | |
|---|---|
| first oder | The first order approximation |
| second oder | The second order approximation |
| third oder | The third order approximation |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kume Alfred and Wood Andrew T.A. (2005). Saddlepoint approximations for the Bingham and Fisher-Bingham normalizing constants. Biometrika, 92(2):465-476

### See Also

kent.logcon, rfb, kent.mle, rbingham

### Examples

```
p <- 3  ;  k <- 1
0.5 * p * log(2 * pi) - (p/2 - 1) * log(k) + log( besselI(k, p/2 - 1, expon.scaled = TRUE) ) + k
## normalising constant of the
## von Mises-Fisher distribution
fb.saddle( c(0, k, 0), c(0, 0, 0) ) ## saddlepoint approximation

## Normalising constant of the Kent distribution
fb.saddle( c(0, 10, 0), c(0, -2, 2) )
kent.logcon(10, 2)
```

---

## Score test for many simple CIPC and SMPL regressions

*Score test for many simple CIPC and SPML regressions*

---

### Description

Score test for many simple CIPC and SPML regressions.

### Usage

```
score.cipc(y, X, rads = TRUE, tol = 1e-06)
score.spml(y, X, rads = TRUE, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| `y` | The dependent variable, a numerical vector, it can be in radians or degrees. |
| `X` | A matrix with many numerical independent variables. |
| `rads` | If the dependent variable is expressed in rads, this should be TRUE and FALSE otherwise. |
| `tol` | The tolerance value to terminate the Newton-Raphson algorithm in the null model (CIPC without covariates). |

### Details

The score test uses the first derivative (score function) of the regression log-likelihood and it is asymptotically correct. So, this function requires sample sizes or at least 1,000 observations. The CIPC is basically the Wrapped Cauchy distribution (Tsagris and Alzeley, 2024) and SPML is the bivariate projected normal (Presnell et al., 1998).

### Value

A matrix with two columns, the test statistic and its associated p-value.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Presnell B., Morrison S. P. and Littell R. C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

## See Also

`cipc.reg`, `spml.reg`, `cipc.mle`, `spml.mle`,

## Examples

```
y <- rcipc(500, omega = 2, g = 5)
x <- matrix( rnorm(500 * 10), ncol = 10 )
a <- score.cipc(y, x)
```

---

`Simulation from a Bingham distribution using any symmetric matrix A`
*Simulation from a Bingham distribution using any symmetric matrix A*

---

## Description

It simulates random values from a Bingham distribution with any given symmetric matrix.

## Usage

```
rbingham(n, A)
```

## Arguments

| | |
|---|---|
| `n` | The sample size. |
| `A` | A symmetric matrix. |

## Details

The eigenvalues are fist calcualted and then Chris Fallaize and Theo Kypraio's code (f.rbing) is used. The resulting simulated data anre then right multiplied by the eigenvectors of the matrix A.

## Value

A matrix with the siumlated data.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

Kent J. T., Ganeiber A. M. and Mardia K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. Journal of Computational and Graphical Statistics, 27(2): 291–301.

Kent J.T., Ganeiber A.M. and Mardia K.V. (2013). A new method to simulate the Bingham and related distributions in directional data analysis with applications. http://arxiv.org/pdf/1310.8110v1.pdf

Fallaize C. J. and Kypraios T. (2016). Exact bayesian inference for the Bingham distribution. Statistics and Computing, 26(1): 349–360. http://arxiv.org/pdf/1401.2894v1.pdf

### See Also

f.rbing, rfb, rvmf, rkent

### Examples

```
A <- cov(iris[, 1:3])
x <- rbingham(100, A)
```

---

Simulation from a Matrix Fisher distribution on SO(3)
*Simulation from a Matrix Fisher distribution on SO(3)*

---

### Description

It simulates random samples (rotation matrices) from a Matrix Fisher distribution with any given parameter matrix, F (3x3).

### Usage

```
rmatrixfisher(n, F)
```

### Arguments

| | |
|---|---|
| n | the sample size. |
| F | An arbitrary 3x3 matrix. |

### Details

Firstly corresponding Bingham parameter A is determined for a given Matrix Fisher parameter F using John Kent et al.'s (2013) algorithm and then Bingham samples for parameter A are generated using rbingham code. Finally convert Bingham samples to Matrix Fisher samples according to the Kent (2013) transformation.

### Value

An array with simulated rotation matrices.

## Author(s)

Anamul Sajib and Chris Fallaize.

R implementation and documentation: Anamul Sajib <sajibstat@du.ac.bd> and Chris Fallaize.

## References

Kent J. T., Ganeiber A. M. and Mardia K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. Journal of Computational and Graphical Statistics, 27(2): 291–301.

Kent J.T., Ganeiber A.M. and Mardia K.V. (2013). A new method to simulate the Bingham and related distributions in directional data analysis with applications. http://arxiv.org/pdf/1310.8110v1.pdf

## See Also

matrixfisher.mle

## Examples

```
F <- matrix( c(85, 11, 41, 78, 39, 60, 43, 64, 48), ncol = 3) / 10   ### An arbitrary F matrix
a <- rmatrixfisher(10, F)
```

---

`Simulation of random values from a Bingham distribution`

*Simulating from a Bingham distribution*

---

## Description

It simulates from a Bingham distribution using the code suggested by Kent et al. (2013).

## Usage

```
f.rbing(n, lam, fast = FALSE)
```

## Arguments

| | |
|---|---|
| `n` | Sample size. |
| `lam` | Eigenvalues of the diagonal symmetric matrix of the Bingham distribution. |
| `fast` | If you want a fast, efficient simulation set this to TRUE. |

## Details

The user must have calculated the eigenvalues of the diagonal symmetric matrix of the Bingham distribution. The function accepts the q-1 eigenvalues only. This means, that the user must have subtracted the lowest eigenvalue from the rest and give the non zero ones. The function uses rejection sampling and it was written by Chris Fallaize and Theo Kypraios (University of Nottingham) and kindly offered. Any questions on the code can be addressed to one of the two aforementioned people. It is slightly different than the one Kent et al. (2013) suggests.

### Value

A list including:

| | |
|---|---|
| X | The simulated data. |
| avtry | The estimate of M in the rejection sampling. The average number of simulated values before a value is accepted. If the argument fast is set to TRUE this information will not appear. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent J. T., Ganeiber A. M. and Mardia K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. Journal of Computational and Graphical Statistics, 27(2): 291–301.

Kent J.T., Ganeiber A.M. and Mardia K.V. (2013). A new method to simulate the Bingham and related distributions in directional data analysis with applications. http://arxiv.org/pdf/1310.8110v1.pdf

Fallaize C. J. and Kypraios T. (2016). Exact bayesian inference for the Bingham distribution. Statistics and Computing, 26(1): 349–360. http://arxiv.org/pdf/1401.2894v1.pdf

### See Also

rfb, rvmf, rbingham, rkent, link{rsop}

### Examples

```
x <- f.rbing( 100, c(1, 0.6, 0.1) )
x
```

---

## Simulation of random values from a mixture of rotationally symmetric distributions

*Simulation of random values from a mixture of rotationally symmetric distributions*

### Description

The function simulates random values simulation from a given mixture of rotationally symmetric distributions.

## Usage

```
rmixvmf(n, probs, mu, k)
rmixspcauchy(n, probs, mu, k)
rmixpkbd(n, probs, mu, k)
```

## Arguments

| | |
|---|---|
| `n` | The sample size. |
| `probs` | This is avector with the mixing probability of each group. |
| `mu` | A matrix with the mean direction of each group. |
| `k` | A vector with the concentration parameter of each group. |

## Details

The function simulates random values simulation from a given mixture of von Mises-Fisher, spherical Cauchy or Poisson kernel-based distributions.

## Value

A list including:

| | |
|---|---|
| `id` | An indicator of the group of each simulated vector. |
| `x` | A matrix with the simulated data. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Kurt Hornik and Bettina Grun (2014). movMF: An R Package for Fitting Mixtures of von Mises-Fisher Distributions http://cran.r-project.org/web/packages/movMF/vignettes/movMF.pdf

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

## See Also

`mixvmf.mle`, `rvmf`, `bic.mixvmf`

## Examples

```
k <- runif(3, 4, 20)
probs <- c(0.2, 0.5, 0.3)
mu <- matrix(rnorm(9), ncol = 3)
mu <- mu / sqrt( rowSums(mu^2) )
x <- rmixvmf(200, probs, mu, k)$x
bic.mixvmf(x, 5)
```

`Simulation of random values from a spherical Fisher-Bingham distribution`
*Simulation of random values from a spherical Fisher-Bingham distribution*

### Description

Simulation of random values from a spherical Fisher-Bingham distribution.

### Usage

```
rfb(n, k, m, A)
```

### Arguments

| | |
|---|---|
| `n` | The sample size. |
| `k` | The concentraion parameter (Fisher part). It has to be greater than 0. |
| `m` | The mean direction (Fisher part). |
| `A` | A symmetric matrix (Bingham part). |

### Details

Random values from a spherical Fisher-Bingham distribution are generated. This functions included the option of simulating from a Kent distribution also.

### Value

A matrix with the simulated data.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kent J. T., Ganeiber A. M. and Mardia K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. Journal of Computational and Graphical Statistics, 27(2): 291–301.

Kent J.T., Ganeiber A.M. and Mardia K.V. (2013). A new method to simulate the Bingham and related distributions in directional data analysis with applications. http://arxiv.org/pdf/1310.8110v1.pdf

Fallaize C. J. and Kypraios T. (2016). Exact bayesian inference for the Bingham distribution. Statistics and Computing, 26(1): 349–360. http://arxiv.org/pdf/1401.2894v1.pdf

### See Also

rbingham, rvmf, rkent, f.rbing

### Examples

```
k <- 15
mu <- rnorm(3)
mu <- mu / sqrt( sum(mu^2) )
A <- cov(iris[, 1:3])
x <- rfb(50, k, mu, A)
vmf.mle(x) ## fits a von Mises-Fisher distribution to the simulated data
## Next we simulate from a Kent distribution
A <- diag( c(-5, 0, 5) )
n <- 100
x <- rfb(n, k, mu, A) ## data follow a Kent distribution
kent.mle(x) ## fits a Kent distribution
vmf.mle(x) ## fits a von Mises-Fisher distribution
A <- diag( c(5, 0, -5) )
n <- 100
x <- rfb(n, k, mu, A) ## data follow a Kent distribution
kent.mle(x) ## fits a Kent distribution
vmf.mle(x) ## fits a von Mises-Fisher distribution
```

---

## Simulation of random values from a spherical Kent distribution

*Simulation of random values from a spherical Kent distribution*

---

### Description

Simulation of random values from a spherical Kent distribution.

### Usage

```
rkent(n, k, m, b)
```

### Arguments

| | |
|---|---|
| n | The sample size. |
| k | The concentraion parameter $\kappa$. It has to be greater than 0. |
| m | The mean direction (Fisher part). |
| b | The ovalness parameter, $\beta$. |

### Details

Random values from a Kent distribution on the sphere are generated. The function generates from a spherical Kent distribution using rfb with an arbitrary mean direction and then rotates the data to have the desired mean direction.

### Value

A matrix with the simulated data.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Kent J. T., Ganeiber A. M. and Mardia K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. Journal of Computational and Graphical Statistics, 27(2): 291–301.

Kent J.T., Ganeiber A.M. and Mardia K.V. (2013). A new method to simulate the Bingham and related distributions in directional data analysis with applications. http://arxiv.org/pdf/1310.8110v1.pdf

### See Also

rfb, rbingham, rvmf, f.rbing

### Examples

```
k <- 15
mu <- rnorm(3)
mu <- mu / sqrt( sum(mu^2) )
A <- diag( c(-5, 0, 5) )
x <- rfb(500, k, mu, A)
kent.mle(x)
y <- rkent(500, k, mu, A[3, 3])
kent.mle(y)
```

---

## Simulation of random values from rotationally symmetric distributions

*Simulation of random values from rotationally symmetric distributions*

---

### Description

Simulation of random values from rotationally symmetric distributions. The data can be spherical or hyper-spherical.

### Usage

```
rvmf(n, mu, k)
riag(n, mu)
rspcauchy(n, mu, rho)
rpkbd(n, mu, rho)
```

### Arguments

| | |
|---|---|
| `n` | The sample size. |
| `mu` | A unit vector showing the mean direction for the von Mises-Fisher or the spherical Cauchy distribution. The mean vector of the Independent Angular Gaussian distribution does not have to be a unit vector. |
| `k` | The concentration parameter ($\kappa$) of the von Mises-Fisher distribution. If $\kappa = 0$, random values from the spherical uniform will be drwan. |
| `rho` | The $\rho$ parameter of the spherical Cauchy or the Poisson kernel-based distribution. |

### Details

The von Mises-Fisher uses the rejection smapling suggested by Wood (1994). For the Independent Angular Gaussian, values are generated from a multivariate normal distribution with the given mean vector and the identity matrix as the covariance matrix. Then each vector becomes a unit vector. For the spherical Cauchy distribution the algortihm is described in Kato and McCullagh (2020) and for the Poisson kernel-based distribution, it is described in Sablica, Hornik and Leydold (2023).

### Value

A matrix with the simulated data.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris `<mtsagris@uoc.gr>` and Giorgos Athineou <gioathineou@gmail.com>.

### References

Wood A.T.A. (1994). Simulation of the von Mises Fisher distribution. Communications in Statistics-Simulation and Computation, 23(1): 157–164.

Dhillon I. S. and Sra S. (2003). Modeling data using directional distributions. Technical Report TR-03-06, Department of Computer Sciences, The University of Texas at Austin. http://citeseerx.ist.psu.edu/viewdoc/download?

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248. https://arxiv.org/pdf/1510.07679.pdf

Sablica L., Hornik K. and Leydold J. (2023). Efficient sampling from the PKBD distribution. Electronic Journal of Statistics, 17(2): 2180–2209.

### See Also

`vmf.mle`, `iag.mle` `rfb`, `racg`, `rvonmises`, `rmixvmf`

### Examples

```
m <- rnorm(4)
m <- m/sqrt(sum(m^2))
x <- rvmf(100, m, 25)
m
vmf.mle(x)
```

---

## Simulation of random values from some circular distributions

*Simulation of random values from some circular distributions*

---

### Description

Simulation of random values from some circular distributions.

### Usage

```
rvonmises(n, m, k, rads = TRUE)
rwrapcauchy(n, m, rho, rads = TRUE)
rspml(n, mu, rads = TRUE)
rcircbeta(n, m, a, b, rads = TRUE)
rcircpurka(n, m, a, rads = TRUE)
rcircexp(n, lambda, rads = TRUE)
rcipc(n, mu = NULL, omega, g, rads = TRUE)
rgcpc(n, mu = NULL, omega, g, rho, rads = TRUE)
```

### Arguments

| | |
|---|---|
| n | The sample size. |
| m | The mean angle expressed in radians or degrees. |
| mu | The mean vector of the SPML, CIPC and GCPC in $R^2$. For the CIPC and GCPC, if this argument is not given, then the omega and g must be given. |
| omega | The location parameter for the CIPC and the GCPC expressed in radians or degrees. |
| k | The concentration parameter of the von Mises distribution. If k is zero the sample will be generated from the uniform distribution over $(0, 2\pi)$. |
| g | The norm of the mean vector for the CIPC and GCPC, if `omega` is given instead of `mu`. |
| rho | For the wrapped Cauchy distribution, this is the $\rho$ parameter. For the GCPC distribution this is the eigenvalue parameter, or covariance determinant. |
| a | The $\alpha$ parameter of the beta distribution. |
| b | The $\beta$ parameter of the beta distribution. |
| lambda | The $\lambda$ parameter of the circular (wrapped) exponential distribution. |
| rads | If the mean angle is expressed in radians, this should be TRUE and FALSE otherwise. The simulated data will be expressed in radians or degrees depending on what the mean angle is expressed. |

## Details

For the von Mises distribution, the mean direction is transformed to the Euclidean coordinates (i.e. unit vector) and then the `rvmf` function is employed. It uses a rejection smapling as suggested by Andrew Wood in 1994. We have mentioned the description of the algorithm as we found it in Dhillon and Sra in 2003. Finally, the data are transformed to radians or degrees.

For the wrapped Cauchy and wrapped exponential distributions the function generates Cauchy or exponential values $x$ and then wrapps them around the circle $x = x(mod 2\pi)$. For the circular beta the function has some extra steps (see Zheng Sun's master thesis).

For the CIPC and GCPC distributions, data are generated from the bivariate Cauchy distribution, normalized to have unit norm and then transformed to angles.

## Value

A vector with the simulated data.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Wood A.T.A. (1994). Simulation of the von Mises Fisher distribution. Communications in Statistics-Simulation and Computation, 23(1): 157-164.

Dhillon I.S. and Sra S. (2003). Modeling data using directional distributions. Technical Report TR-03-06, Department of Computer Sciences, The University of Texas at Austin. http://citeseerx.ist.psu.edu/viewdoc/download?d

Zheng Sun (2006). Comparing measures of fit for circular distributions. Master thesis, University of Victoria. https://dspace.library.uvic.ca/bitstream/handle/1828/2698/zhengsun_master_thesis.pdf;sequence=1

Lai M. (1994). Some results in the statistical analysis of directional data. Master thesis, University of Hong Kong.

Presnell B., Morrison S.P. and Littell R.C. (1998). Projected multivariate linear models for directional data. Journal of the American Statistical Association, 93(443): 1068–1077.

Purkayastha S. (1991). A Rotationally Symmetric Directional Distribution: Obtained through Maximum Likelihood Characterization. The Indian Journal of Statistics, Series A, 53(1): 70–83

Jammalamadaka S.R. and Kozubowski T.J. (2003). A new family of circular models: The wrapped Laplace distributions. Advances and Applications in Statistics, 3(1): 77–103.

## See Also

`circ.summary`, `rvmf`, `racg`

## Examples

```
x <- rvonmises(100, 2, 25, rads = TRUE)
circ.summary(x, rads = TRUE)
```

---

`Simulation of random values from the ESAG distribution`
*Simulation of random values from the ESAG distribution*

---

### Description

Simulation of random values from the ESAG distribution.

### Usage

```
resag(n, mu, gam)
rESAGd(n, mu, gam)
```

### Arguments

| | |
|---|---|
| `n` | A number; how many vectors you want to generate. |
| `mu` | The mean vector the ESAG distribution. |
| `gam` | The $\gamma$ parameters of the ESAG distribution. For the rESAGd this may be NULL in case you want to simulate from the IAG in arbitrady dimensions. |

### Details

A random sample from the ESAG distribution is generated. In case the $\gamma_s$ are zero (or null for the rESAGd), the sample is drawn from the Independent Angular Gaussian (or projected normal). The resag() is designed for the sphere, whereas the rESAGd is designed for the sphere and hyper-sphere.

### Value

An $n \times d$ matrix with the simulated unit vectors.

### Author(s)

Michail Tsagris and Zehao Yu.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Zehao Yu <zehaoy@email.sc.edu>.

### References

Zehao Yu and Xianzheng Huang (2024). A new parameterization for elliptically symmetric angular Gaussian distributions of arbitrary dimension. Electronic Journal of Statististics, 18(1): 301–334.

Paine P.J., Preston S.P., Tsagris M. and Wood A.T.A. (2018). An Elliptically Symmetric Angular Gaussian Distribution. Statistics and Computing, 28(3):689–697.

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

`esag.mle`, `desag`, `spml.mle`, `acg.mle`, `circ.summary`

## Examples

```
m <- colMeans( as.matrix( iris[, 1:3] ) )
y <- resag(1000, m, c(1, 0.5) )
esag.mle(y)
```

---

Simulation of random values from the SESPC distribution
*Simulation of random values from the SESPC distribution*

---

## Description

Simulation of random values from the SESPC distribution

## Usage

```
rsespc(n, mu, theta)
```

## Arguments

| | |
|---|---|
| `n` | A number; how many vectors you want to generate. |
| `mu` | The mean vector the SESPC distribution, a vector in $R^3$. |
| `theta` | The two $\theta$ parameters of the SESPC distribution. |

## Details

A random sample from the SESPC distribution is generated. In case the $\theta_s$ are zero, the sample is drawn from the SIPC (spherical independent projected Cauchy) distribution.

## Value

An $n \times 3$ matrix with the simulated unit vectors.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

`sespc.mle`, `dsespc`

## Examples

```
m <- colMeans( as.matrix( iris[,1:3] ) )
y <- rsespc(1000, m, c(1, 0.5) )
sespc.mle(y)
```

---

`Spherical and hyper-spherical distance correlation`
*Spherical and hyper-spherical distance correlation*

---

## Description

Spherical and hyper-spherical distance correlation.

## Usage

```
spher.dcor(x, y)
```

## Arguments

| | |
|---|---|
| `x` | A matrix with directional data, i.e. unit vectors. |
| `y` | A matrix with directional data, i.e. unit vectors. |

## Details

The distance correlation between two spherical or hyper-spherical variables is computed.

## Value

A list including:

| | |
|---|---|
| `dcov` | The distance covariance. |
| `dvarX` | The distance variance of x. |
| `dvarY` | The distance variance of Y. |
| `dcor` | The distance correlation. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

## References

G.J. Szekely, M.L. Rizzo and N. K. Bakirov (2007). Measuring and Testing Independence by Correlation of Distances. Annals of Statistics, 35(6):2769-2794.

### See Also

circ.dcor

### Examples

```
y <- rvmf(50, rnorm(3), 4)
x <- rvmf(50, rnorm(3), 4)
spher.dcor(x, y)
```

---

## Spherical and hyperspherical median

*Fast calculation of the spherical and hyperspherical median*

---

### Description

It calculates, very fast, the (hyper-)spherical median of a sample.

### Usage

```
mediandir(x)
mediandir_2(x)
```

### Arguments

| | |
|---|---|
| x | The data, a numeric matrix with unit vectors. |

### Details

The "mediandir" employes a fixed poit iterative algorithm stemming from the first derivative (Cabrera and Watson, 1990) to find the median direction as described by Fisher (1985) and Fisher, Lewis and Embleton (1987). In the big samples this is much much faster than "mediandir_2", since the search is based on iterations.

### Value

The median direction.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Fisher N. I. (1985). Spherical medians. Journal of the Royal Statistical Society. Series B, 47(2): 342–348.

Fisher N. I., Lewis T. and Embleton B. J. (1987). Statistical analysis of spherical data. Cambridge university press.

Cabrera J. and Watson G. S. (1990). On a spherical median related distribution. Communications in Statistics-Theory and Methods, 19(6): 1973–1986.

### See Also

nsmedian, vmf.mle, kent.mle

### Examples

```
m <- rnorm(3)
m <- m / sqrt( sum(m^2) )
x <- rvmf(100, m, 10)
mediandir(x)
mediandir_2(x)
nsmedian(x)
```

---

Spherical regression using rotationally symmetric distributions
*Spherical regression using rotationally symmetric distributions*

---

### Description

Spherical regression using rotationally symmetric distributions.

### Usage

```
iag.reg(y, x, con = TRUE, xnew = NULL, tol = 1e-06)
vmf.reg(y, x, con = TRUE, xnew = NULL, tol = 1e-06)
sipc.reg(y, x, con = TRUE, xnew = NULL, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| y | A matrix with 3 columns containing the (unit vector) spherical data. |
| x | The predictor variable(s), they can be continnuous, spherical, categorical or a mix of them. |
| con | Do you want the constant term in the regression? |
| xnew | If you have new data use it, otherwise leave it NULL. |
| tol | A tolerance value to decide when to stop the successive optimaizations. |

### Details

The second parametrization of the projected normal and of the von Mises-Fisher regression (Paine et al., 2020) is applied. The same is true for the SIPC distribution. For more information see the paper by Paine et al. (2020).

### Value

A list including:

| | |
|---|---|
| `loglik` | The log-likelihood of the regression model. |
| `fit` | This is a measure of fit of the estimated values, defined as $\sum_{i=1}^n y_i^T \hat{y}_i$. This appears if the argument "xnew" is NULL. |
| `beta` | The beta coefficients. |
| `seb` | The standard error of the beta coefficients. |
| `ki` | The norm of the fitted values. In the von Mises-Fisher regression this is the concentration parameter of each observation. In the projected normal this are the norms of the fitted values before being projected onto the sphere. This is returned if the argument "xnew" is NULL. |
| `est` | The fitted values of xnew if "xnew" is NULL. If it is not NULL, the fitted values for the "xnew" you supplied will be returned. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`>.

### References

P. J. Paine, S. P. Preston, M. Tsagris and Andrew T. A. Wood (2020). Spherical regression models with general covariates and anisotropic errors. Statistics and Computing, 30(1): 153–165. https://link.springer.com/content/pdf/10.1007

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

### See Also

`esag.reg`, `vmfreg`, `spml.reg`

### Examples

```
y <- rvmf(150, rnorm(3), 5)
a1 <- iag.reg(y, iris[, 4])
a2 <- iag.reg(y, iris[, 4:5])

b1 <- vmf.reg(y, iris[, 4])
b2 <- vmf.reg(y, iris[, 4:5])
```

---

Spherical regression using the ESAG distribution
*Spherical regression using the ESAG distribution*

---

### Description

Spherical regression using the ESAG distribution.

### Usage

```
esag.reg(y, x, con = TRUE, xnew = NULL, lati = 10, longi = 10, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| `y` | A matrix with 3 columns containing the (unit vector) spherical data. |
| `x` | The predictor variable(s), they can be continnuous, spherical, categorical or a mix of them. |
| `con` | Do you want the constant term in the regression? |
| `xnew` | If you have new data use it, otherwise leave it NULL. |
| `lati` | A positive number determing the range of degrees to move left and right from the latitude center. This number and the next determine the grid of points to search for the Q matrix described in Paine et al. (2020). |
| `longi` | A positive number determing the range of degrees to move up and down from the longitude center. This number and the previous determine the grid of points to search for the Q matrix described in Paine et al. (2020). |
| `tol` | A tolerance value to decide when to stop the successive optimizations. |

### Details

The second parametrization of the ESAG regression (Paine et al., 2020) is applied.

### Value

A list including:

| | |
|---|---|
| `loglik` | The log-likelihood of the regression model. |
| `param` | A vector with three numbers. A measure of fit of the estimated values, defined as $\sum_{i=1}^n y_i^T \hat{y}_i$. This appears if the argument "xnew" is NULL. The $\rho \in (0, 1]$ (smallest eigenvalue of the covariance matrix)), and the angle of rotation $psi$. |
| `gam` | The two $\gamma$ parameters. |
| `beta` | The beta coefficients. |
| `seb` | The standard error of the beta coefficients. |
| `est` | The fitted values of xnew if "xnew" is NULL. If it is not NULL, the fitted values for the "xnew" you supplied will be returned. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

P. J. Paine, S. P. Preston, M. Tsagris and Andrew T. A. Wood (2020). Spherical regression models with general covariates and anisotropic errors. Statistics and Computing, 30(1): 153–165. https://link.springer.com/content/pdf/10.1007

### See Also

esag.mle, iag.reg, spml.reg

### Examples

```
y <- resag( 25, rnorm(3), c(1, 1) )
## this is a small example to pass CRAN's check because the default argument values
## of lati and longi require many seconds
a <- esag.reg(y, iris[1:25, 4], lati = 2, longi = 2)
```

---

## Spherical regression using the SESPC distribution

*Spherical regression using the SESPC distribution*

---

### Description

Spherical regression using the SESPC distribution.

### Usage

```
sespc.reg(y, x, con = TRUE, xnew = NULL, lati = 10, longi = 10, tol = 1e-06)
```

### Arguments

| | |
|---|---|
| y | A matrix with 3 columns containing the (unit vector) spherical data. |
| x | The predictor variable(s), they can be continnuous, spherical, categorical or a mix of them. |
| con | Do you want the constant term in the regression? |
| xnew | If you have new data use it, otherwise leave it NULL. |
| lati | A positive number determing the range of degrees to move left and right from the latitude center. This number and the next determine the grid of points to search for the Q matrix described in Tsagris and Alzeley (2024). |
| longi | A positive number determing the range of degrees to move up and down from the longitude center. This number and the previous determine the grid of points to search for the Q matrix described in Tsagris and Alzeley (2024). |
| tol | A tolerance value to decide when to stop the successive optimizations. |

## Details

Regression based on the SESPC distribution (Tsagris and Alzeley, 2025) is applied.

## Value

A list including:

- `loglik` The log-likelihood of the regression model.
- `param` A vector with three numbers. A measure of fit of the estimated values, defined as $\sum_{i=1}^n y_i^T \hat{y}_i$. This appears if the argument "xnew" is NULL. The $\rho \in (0, 1]$ (smallest eigenvalue of the covariance matrix)), and the angle of rotation $psi$.
- `theta` The two $\theta$ parameters.
- `beta` The beta coefficients.
- `seb` The standard error of the beta coefficients.
- `est` The fitted values of xnew if "xnew" is NULL. If it is not NULL, the fitted values for the "xnew" you supplied will be returned.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Tsagris M. and Alzeley O. (2025). Circular and spherical projected Cauchy distributions: A Novel Framework for Circular and Directional Data Modeling. Australian & New Zealand Journal of Statistics, 67(1): 77–103. https://arxiv.org/pdf/2302.02468.pdf

## See Also

esag.mle, iag.reg, spml.reg

## Examples

```
y <- rsespc( 150, rnorm(3), c(1, 1) )
## this is a small example to pass CRAN's check because the default argument values
## of lati and longi require many seconds
a <- sespc.reg(y, iris[, 4], lati = 2, longi = 2)
```

`Spherical-spherical correlation` *Spherical-spherical correlation*

### Description

Correlation between two spherical variables.

### Usage

```
spher.cor(x, y)
```

### Arguments

| | |
|---|---|
| `x` | A spherical variable. A matrix with thre columns, each row is a unit vector. |
| `y` | A spherical variable. A matrix with thre columns, each row is a unit vector. |

### Details

A very similar to the classical correlation is calcualted. In addition, a hypothesis test for no correlation is performed. Note, that this is a squared correlation actually, so negative values will never be returned.

### Value

A vector including:

| | |
|---|---|
| `R-squared` | The value of the squared correlation. |
| `p-value` | The p-value of the no correlation hypothesis testing. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kanti V. Mardia and Peter E. Jupp. Directional statistics, pg. 254–255.

### See Also

`spher.reg`, `vmf.mle`, `circ.cor1`, `circ.cor2`

### Examples

```
x <- rvmf(100, rnorm(3), 10)
y <- rvmf(100, rnorm(3), 10)
spher.cor(x, y)
```

Spherical-spherical regression
*Spherical-Spherical regression*

## Description

Regression when both the dependent and independent variables are spherical.

## Usage

```
spher.reg(y, x, rads = FALSE, xnew = NULL)
```

## Arguments

| | |
|---|---|
| y | The dependent variable; a matrix with either two columns, latitude and longitude, either in radians or in degrees. Alternatively it is a matrix with three columns, unit vectors. |
| x | The dependent variable; a matrix with either two columns, latitude and longitude, either in radians or in degrees. Alternatively it is a matrix with three columns, unit vectors. The two matrices must agree in the scale and dimensions. |
| rads | If the data are expressed in latitude and longitude then it matter to know if they are in radians or degrees. If they are in radians, then this should be TRUE and FALSE otherwise. If the previous argument, euclidean, is TRUE, this one does not matter what its value is. |
| xnew | The new values of some spherical independent variable(s) whose spherical response values you want to predict. If you have no new x values, leave it NULL (default). |

## Details

Spherical regression as proposed by Chang (1986) is implemented. If the estimated rotation matrix has a determinant equal to -1, singular value decomposition is performed and the third unit vector is multiplied by -1.

## Value

A list including:

| | |
|---|---|
| A | The estimated rotation matrix. |
| est | The fitted values in unit vectors, if the argument xnew is not NULL. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Ted Chang (1986). Spherical Regression. Annals of Statistics, 14(3): 907–924.

### See Also

hspher.reg, spher.cor, spml.reg

### Examples

```
mx <- rnorm(3)
mx <- mx/sqrt( sum(mx^2) )
my <- rnorm(3)
my <- my/sqrt( sum(my^2) )
x <- rvmf(100, mx, 15)
A <- rotation(mx, my)
y <- x %*% t(A)
mod <- spher.reg(y, x)
A
mod$A ## exact match, no noise
y <- x %*% t(A)
y <- y + rvmf(100, colMeans(y), 40)
mod <- spher.reg(y, x)
A
mod$A ## noise added, more relistic example
```

---

`Summary statistics for circular data`

*Summary statistics for circular data*

---

### Description

It produces a few summary measures for circular data.

### Usage

```
circ.summary(u, rads = FALSE, fast = FALSE, tol = 1e-07, plot = FALSE)
```

### Arguments

| | |
|---|---|
| `u` | A vector with circular data. |
| `rads` | If the data are in rads, then this should be TRUE, otherwise FALSE. |
| `fast` | A boolean variable to do a faster implementation. |
| `tol` | The tolerance level to stop the Newton-Raphson algorithm for finding kappa. |
| `plot` | If you want to see the data plotted on a cicrle make this TRUE. |

### Details

It returns the circular mean, mean resultant length, variance, standard deviation and concentration parameter. So, basically it returns the estimated values of the parameters of the von Mises distribution.

### Value

If fast = FALSE a list including all the following. If fast = TRUE less items are returned.

| | |
|---|---|
| `mesos` | The circular mean direction. |
| `confint` | The 95% confidence interval for the circular mean direction. |
| `kappa` | The concentration parameter. |
| `MRL` | The mean resultant length. |
| `circvariance` | The circular variance. |
| `circstd` | The circular standard deviation. |
| `loglik` | The log-likelihood of the fitted von Mises distribution. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

`spml.mle`, `rvonmises`, `vm.kde`, `vmf.mle`, `group.vm`, `hcf.circaov`

### Examples

```
x <- rvonmises(50, 2.5, 15, rads = TRUE)
circ.summary(x, rads = TRUE, plot = TRUE)
```

---

## Summary statistics for grouped circular data

*Summary statistics for grouped circular data*

---

### Description

It produces a few summary measures for grouped circular data.

## Usage

```
group.vm(group, fi, rads = FALSE)
```

## Arguments

| | |
|---|---|
| `group` | A matrix denoting the classes. Each row consists of two numbers, the lower and upper points of each class. |
| `fi` | The frequency of each class of data. |
| `rads` | If the data are in rads, then this should be TRUE, otherwise FALSE. |

## Details

It returns the circular mean, mean resultant length, variance, standard deviation and concentration parameter. So, basically it returns the estimated values of the parameters of the von Mises distribution. The mena resultant length though is group corrected.

## Value

A list including:

| | |
|---|---|
| `mesos` | The circular mean direction. |
| `confint` | The 95% confidence interval for the circular mean direction. |
| `kappa` | The concentration parameter. |
| `MRL` | The mean resultant length. |
| `circvariance` | The circular variance. |
| `circstd` | The circular standard deviation. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

## References

Pewsey Arthur, Markus Neuhauser and Graeme D. Ruxton (2013). Circular statistics in R. Oxford University Press.

Mardia K. V. and Jupp P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

## See Also

`circ.summary`, `rvonmises`, `vm.kde`

### Examples

```
x <- rvonmises(200, 3, 10)
a <- circ.summary(x, rads = TRUE, plot = FALSE)
group <- seq(min(x) - 0.1, max(x) + 0.1, length = 6)
y <- cut(x, breaks = group, length = 6)
group <- matrix( c( group[1], rep(group[2:5], each = 2), group[6]), ncol = 2, byrow = TRUE)
fi <- as.vector( table(y) )
b <- group.vm(group, fi, rads = TRUE)
a
b
```

---

`Test for a given mean direction`

## *Test for a given mean direction*

---

### Description

A log-likelihood ratio test for testing whether the sample mena direction is equal to some predefined one.

### Usage

```
meandir.test(x, mu, B = 999)
```

### Arguments

| | |
|---|---|
| `x` | A matrix with the data, unit vectors. |
| `mu` | A unit vector with the hypothesized mean direction. |
| `B` | A number either 1, so no bootstrap calibration is performed or more than 1, so bootstrap calibration is performed. |

### Details

The log-likelihood ratio test is employed.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degrees of freedom of the test. If bootstrap was employed this is "NA". |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

vmf.mle, kent.mle, rayleigh

### Examples

```
mu <- rnorm(5)
mu <- mu / sqrt( sum(mu^2) )
x <- rvmf(100, mu, 10)
meandir.test(x, mu, 1)
meandir.test(x, mu, 499)
```

---

## Test for equality of concentration parameters for spherical data

*Test for equality of concentration parameters for spherical data*

---

### Description

This tests the equality of concentration parameters for spherical data only.

### Usage

```
spherconc.test(x, ina)
```

### Arguments

| | |
|---|---|
| x | A matrix with the data in Euclidean coordinates, i.e. unit vectors |
| ina | A variable indicating the groupings of the observations. |

### Details

The test is designed for spherical data only.

### Value

A list including:

| | |
|---|---|
| mess | A message stating the value of the mean resultant and which test statistic was used, U1, U2 or U3. |
| res | A vector containing the test statistic and its p-value. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Kanti V. Mardia and Peter E. Jupp. Directional statistics, pg. 226–227.

### See Also

het.aov, lr.aov, embed.aov, hcf.aov, conc.test, sphereplot

### Examples

```
x <- rvmf(100, rnorm(3), 15)
ina <- rep(1:4, each = 25)
spherconc.test(x, ina)
```

---

## Test of equality of the concentration parameters for circular data

*A test for testing the equality of the concentration parameter among g samples, where g >= 2 for ciruclar data*

---

### Description

A test for testing the equality of the concentration parameter among g samples, where g >= 2 for ciruclar data.

### Usage

```
conc.test(u, ina, rads = FALSE)
```

### Arguments

| | |
|---|---|
| u | A numeric vector containing the values of all samples. |
| ina | A numerical variable or factor indicating the groups of each value. |
| rads | If the data are in radians this should be TRUE and FALSE otherwise. |

### Details

This test works for circular data.

### Value

A list including:

mess    A message informing the use of the test statistic used.

res    A numeric vector containing the value of the test statistic and its associated p-value.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris `<mtsagris@uoc.gr>` and Giorgos Athineou <gioathineou@gmail.com>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

### See Also

`embed.circaov`, `hcf.circaov`, `lr.circaov`, `het.circaov`

### Examples

```
x <- rvonmises(100, 2.4, 15)
ina <- rep(1:4,each = 25)
conc.test(x, ina, rads = TRUE)
```

---

## The k-nearest neighbours using the cosinus distance

*The k-nearest neighbours using the cosinus distance*

---

### Description

The k-nearest neighbours using the cosinus distance.

### Usage

```
cosnn(xnew, x, k = 5, index = FALSE, rann = FALSE)
```

### Arguments

xnew    The new data whose k-nearest neighbours are to be found.

x    The data, a numeric matrix with unit vectors.

k    The number of nearest neighbours, set to 5 by default. It can also be a vector with many values.

index    If you want the indices of the closest observations set this equal to TRUE.

rann    If you have large scale datasets and want a faster k-NN search, you can use kd-trees implemented in the R package "RANN". In this case you must set this argument equal to TRUE.

### Details

The shortest distances or the indices of the k-nearest neighbours using the cosinus distance are returned.

### Value

A matrix with the shortest distance of each xnew from x, if index is FALSE, or the indices of the nearest neighbours of each xnew from x, if index is TRUE.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Tsagris M. and Alenazi A. (2019). Comparison of discriminant analysis methods on the sphere. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(4): 467–491.

### See Also

dirknn, dirknn.tune

### Examples

```
xnew <- rvmf(10, rnorm(3), 5)
x <- rvmf(50, rnorm(3), 5)
a <- cosnn(xnew, x, k = 5)
b <- cosnn(xnew, x, k = 5, index = TRUE)
```

---

Transform unit vectors to angular data

## *Transform unit vectors to angular data*

---

### Description

Transform unit vectors to angular data.

### Usage

```
etoa(x)
```

### Arguments

| | |
|---|---|
| x | A numerical matrix with directional data, i.e. unit verctors. |

### Details

from the Euclidean coordinates the data are mapped to angles, expressed in rads.

### Value

A list including:

mu	A matrix with angles. The number of columns is one less than that of the original matrix.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

https://en.wikipedia.org/wiki/N-sphere#Spherical_coordinates

### See Also

vmnb.pred

### Examples

```
x <- rvmf(10, rnorm(3), 5)
y <- etoa(x)
```

---

## Tuning of the bandwidth parameter in the von Mises kernel

*Tuning of the bandwidth parameter in the von Mises kernel for circular data*

---

### Description

Tuning of the bandwidth parameter in the von Mises kernel for circular data. Cross validation is used.

### Usage

```
vmkde.tune(u, low = 0.1, up = 1, rads = TRUE)
```

### Arguments

| | |
|---|---|
| u | The data, a numerical vector. |
| low | The lower value of h to search. |
| up | The lower value of h to search. |
| rads | If the data are in radians this should be TRUE and FALSE otherwise. |

## Details

Tuning of the bandwidth parameter in the von Mises kernel for circula data via cross validation. The procedure is fast because an optimiser is used.

## Value

A vector including two elements:

`Optimal h` The best H found.

`cv` The value of the maximised pseudo-likelihood.

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou <gioathineou@gmail.com>.

## References

Taylor C.C. (2008). Automatic bandwidth selection for circular density estimation. Computational Statistics & Data Analysis, 52(7), 3493–3500.

Wand M.P. and Jones M.C. (1994). Kernel smoothing. CrC Press.

## See Also

vm.kde, vmfkde.tune, vmf.kde

## Examples

```
u <- rvonmises(100, 2.4, 10, rads = TRUE)
vmkde.tune(u)
```

---

# Tuning of the bandwidth parameter in the von Mises-Fisher kernel

*Tuning of the bandwidth parameter in the von Mises-Fisher kernel for (hyper-)spherical data*

---

## Description

Tuning of the bandwidth parameter in the von Mises-Fisher kernel for (hyper-)spherical data whit cross validation.

## Usage

```
vmfkde.tune(x, low = 0.1, up = 1)
```

### Arguments

| | |
|---|---|
| `x` | A matrix with the data in Euclidean cordinates, i.e. unit vectors. |
| `low` | The lower value of the bandwdith to search. |
| `up` | The upper value of the bandwdith to search. |

### Details

Fast tuning of the bandwidth parameter in the von Mises-Fisher kernel for (hyper-)spherical data via cross validation.

### Value

A vector including two elements:

| | |
|---|---|
| `Optimal h` | The best H found. |
| `cv` | The value of the maximised pseudo-likelihood. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Garcia P.E. (2013). Exact risk improvement of bandwidth selectors for kernel density estimation with directional data. Electronic Journal of Statistics, 7, 1655–1685.

Wand M.P. and Jones M.C. (1994). Kernel smoothing. Crc Press.

### See Also

`vmf.kde`, `vmf.kerncontour`, `vm.kde`, `vmkde.tune`

### Examples

```
x <- rvmf(100, rnorm(3), 15)
vmfkde.tune(x)
```

---

Tuning of the k-NN algorithm using the arc cosinus distance
*k-NN algorithm using the arc cosinus distance. Tuning the k neigbours*

---

### Description

It estimates the percentage of correct classification via an m-fold cross validation.

### Usage

```
dirknn.tune(ina, x, k = 2:10, mesos = TRUE, nfolds = 10, folds = NULL,
parallel = FALSE, stratified = TRUE, seed = NULL, rann = FALSE, graph = FALSE)
```

### Arguments

| | |
|---|---|
| `x` | The data, a numeric matrix with unit vectors. |
| `ina` | A variable indicating the groups of the data x. |
| `nfolds` | How many folds to create? |
| `k` | A vector with the number of nearest neighbours to consider. |
| `mesos` | A boolean variable used only in the case of the non standard algorithm (type="NS"). Should the average of the distances be calculated (TRUE) or not (FALSE)? If it is FALSE, the harmonic mean is calculated. |
| `folds` | Do you already have a list with the folds? If not, leave this NULL. |
| `parallel` | If you want the standard -NN algorithm to take place in parallel set this equal to TRUE. |
| `stratified` | Should the folds be created in a stratified way? i.e. keeping the distribution of the groups similar through all folds? |
| `seed` | If seed is TRUE, the results will always be the same. |
| `rann` | If you have large scale datasets and want a faster k-NN search, you can use kd-trees implemented in the R package "RANN". In this case you must set this argument equal to TRUE. |
| `graph` | If this is TRUE a graph with the results will appear. |

### Details

The standard algorithm is to keep the k nearest observations and see the groups of these observations. The new observation is allocated to the most frequent seen group. The non standard algorithm is to calculate the classical mean or the harmonic mean of the k nearest observations for each group. The new observation is allocated to the group with the smallest mean distance.

We have made an eficient (not very much efficient though) memory allocation. Even if you have hundreds of thousands of observations, the computer will not clush, it will only take longer. Instead of calculate the distance matrix once in the beginning we calcualte the distances of the out-of-sample observations from the rest. If we calculated the distance matrix in the beginning, once, the resulting matrix could have dimensions thousands by thousands. This would not fit into the memory. If you have a few hundres of observations, the runtime is about the same (maybe less, maybe more) as calculating the distance matrix in the first place.

### Value

A list including:

- `per` — The average percent of correct classification across the neighbours.
- `percent` — The estimated (optimal) percent of correct classification.
- `runtime` — The run time of the algorithm. A numeric vector. The first element is the user time, the second element is the system time and the third element is the elapsed time.

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Tsagris M. and Alenazi A. (2019). Comparison of discriminant analysis methods on the sphere. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(4), 467–491.

### See Also

dirknn, dirda, mixvmf.mle

### Examples

```
k <- runif(4, 4, 20)
prob <- c(0.2, 0.4, 0.3, 0.1)
mu <- matrix(rnorm(16), ncol = 4)
mu <- mu / sqrt( rowSums(mu^2) )
da <- rmixvmf(200, prob, mu, k)
x <- da$x
ina <- da$id
dirknn.tune(ina, x, k = 2:6, nfolds = 5, mesos = TRUE)
dirknn.tune(ina, x, k = 2:6, nfolds = 10, mesos = TRUE)
```

---

## Tuning of the k-NN regression

*Tuning of the k-NN regression with Euclidean or (hyper-)spherical response and or predictor variables*

---

### Description

Tuning of the k-NN regression with Euclidean or (hyper-)spherical response and or predictor variables. It estimates the percentage of correct classification via an m-fold cross valdiation. The bias is estimated as well using the algorithm suggested by Tibshirani and Tibshirani (2009) and is subtracted.

## Usage

```
knnreg.tune(y, x, nfolds = 10, A = 10, ncores = 1, res = "eucl",
estim = "arithmetic", folds = NULL, seed = NULL, graph = FALSE)
```

## Arguments

| | |
|---|---|
| `y` | The currently available data, the response variables values. A matrix with either euclidean (univariate or multivariate) or (hyper-)spherical data. If you have a circular response, say u, transform it to a unit vector via (cos(u), sin(u)). |
| `x` | The currently available data, the predictor variables values. A matrix with either euclidean (univariate or multivariate) or (hyper-)spherical data. If you have a circular response, say u, transform it to a unit vector via (cos(u), sin(u)). |
| `nfolds` | How many folds to create? |
| `A` | The maximum number of nearest neighbours, set to 10 by default, starting from the 2nd nearest neighbor. |
| `ncores` | How many cores to use. This is taken into account only when the predictor variables are spherical. |
| `res` | The type of the response variable. If it is Euclidean, set this argument equal to "res". If it is a unit vector set it to res="spher". |
| `estim` | Once the k observations whith the smallest distance are discovered, what should the prediction be? The arithmetic average of the corresponding y values be used estim="arithmetic" or their harmonic average estim="harmonic". |
| `folds` | Do you already have a list with the folds? If not, leave this NULL. |
| `seed` | You can specify your own seed number here or leave it NULL. |
| `graph` | If this is TRUE a graph with the results will appear. |

## Details

Tuning of the k-NN regression with Euclidean or (hyper-)spherical response and or predictor variables. It estimates the percentage of correct classification via an m-fold cross valdiation. The bias is estimated as well using the algorithm suggested by Tibshirani and Tibshirani (2009) and is subtracted. The sum of squares of prediction is used in the case of Euclidean responses. In the case of spherical responses the $\sum_i \hat{y}_i^T y_i$ is calculated.

## Value

A list including:

| | |
|---|---|
| `crit` | The value of the criterion to minimise/maximise for all values of the nearest neighbours. |
| `best_k` | The best value of the nearest neighbours. |
| `performance` | The bias corrected optimal value of the criterion, along wit the estimated bias. For the case of Euclidean reponse this will be higher than the crit and for the case or spherical responses it will be lower than crit. |
| `runtime` | The run time of the algorithm. A numeric vector. The first element is the user time, the second element is the system time and the third element is the elapsed time. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### See Also

knn.reg, spher.reg, dirknn.tune

### Examples

```
y <- iris[, 1]
x <- iris[, 2:4]
x <- x/ sqrt( rowSums(x^2) )  ## Euclidean response and spherical predictors
knnreg.tune(y, x, A = 5, res = "eucl", estim = "arithmetic")

y <- iris[, 1:3]
y <- y/ sqrt( rowSums(y^2) )  ## Spherical response and Euclidean predictor
x <- iris[, 2]
knnreg.tune(y, x, A = 5, res = "spher", estim = "arithmetic")
```

---

## Two sample location test for (hyper-)spherical data

*Two sample location test for (hyper-)spherical data*

---

### Description

Two sample location test for (hyper-)spherical data.

### Usage

```
spcauchy2test(y1, y2, B = 1)
pkbd2test(y1, y2, B = 1)
vmf2test(y1, y2, B = 1)
sp2(y1, y2, tol = 1e-6)
pk2(y1, y2, tol = 1e-6)
vmf2(y1, y2, tol = 1e-6)
```

### Arguments

| | |
|---|---|
| y1 | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| y2 | A matrix with the data in Euclidean coordinates, i.e. unit vectors. |
| B | The number of bootstraps to perform. |
| tol | The tolerance value at which to terminate the iterations. |

## Details

A log-likelihood ratio based test for the equality of two location parameters, assuming that the data in each group follow the spherical Cauchy of the Poisson kernel-based distribution. Bootstrap is also offered.

For the von Mises-Fisher distribution we do the same, but for the mean direction.

The functions sp2() and pk2() estimate the common location of the two groups assuming unequal concentration parameters. These functions are used the compute the log-likelihood under the null hypothesis. So does the function vmf2(), but the mean direction.

## Value

The result of the spcauchy2test(), pkbd2test() and vmf2test() functions is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degree(s) of freedom of the test. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

The result of the sp2(), pk2() and vmf2 functions is a list including:

| | |
|---|---|
| `mu` | The common location parameter, for both samples, under the null hypothesis. |
| `rho1` | The concentration parameter of the first group, asusming a common location parameter. |
| `rho2` | The concentration parameter of the second group, asusming a common location parameter. |
| `kappa11` | The concentration parameter (assuming the von Mises-Fisher distribution) of the first group, asusming a common location parameter. |
| `kappa2` | The concentration parameter (assuming the von Mises-Fisher distribution) of the second group, asusming a common location parameter. |
| `loglik` | The log-likelihood of the whole sample, asusming a common location (or mean direction) parameter. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Kato S. and McCullagh P. (2020). Some properties of a Cauchy family on the sphere derived from the Mobius transformations. Bernoulli, 26(4): 3224–3248.

Golzy M. and Markatou M. (2020). Poisson kernel-based clustering on the sphere: convergence properties, identifiability, and a method of sampling. Journal of Computational and Graphical Statistics, 29(4): 758–770.

Tsagris M., Papastamoulis P. and Kato S. (2025). Directional data analysis using the spherical Cauchy and the Poisson kernel-based distribution. Statistics and Computing, 35:51.

### See Also

het.boot, het.aov

### Examples

```
mu <- rvmf(2, rnorm(5), 3)
y1 <- rspcauchy(60, mu[1, ], 0.4)
y2 <- rspcauchy(30, mu[2, ], 0.8)
spcauchy2test(y1, y2)
```

---

## Two sample location test for circular data under the GCPC distribution

*Two sample location test for circular data under the GCPC distribution.*

---

### Description

Two sample location test for circular data under the GCPC distribution.

### Usage

```
gcpc.means.test(u1, u2, rads = FALSE)
```

### Arguments

| | |
|---|---|
| u1 | A numeric vector containing circular data for the first sample. |
| u2 | A numeric vector containing circular data for the second sample. |
| rads | If the data are in radians, this should be TRUE and FALSE otherwise. |

### Details

The log-likelihood ratio test compares the location parameter of two independent samples, assuming that both samples are drawn from populations that follow the GCPC distribution.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | The degree(s) of freedom of the test. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr>.

### References

Mardia, K. V. and Jupp, P. E. (2000). Directional statistics. Chicester: John Wiley & Sons.

Rumcheva P. and Presnell B. (2017). An improved test of equality of mean directions for the Langevin-von Mises-Fisher distribution. Australian & New Zealand Journal of Statistics, 59(1): 119–135.

Alzeley O. and Tsagris M. (2026). On the generalized circular projected Cauchy distribution. https://arxiv.org/abs/2603.04030.

### See Also

`hcf.circaov`, `spcauchy2test`

### Examples

```
u1 <- rgcpc(50, omega = 2, g = 5, rho = 0.5, rads = TRUE)
u2 <- rgcpc(50, omega = 2, g = 10, rho = 5, rads = TRUE)
gcpc.means.test(u1, u2, rads = TRUE)
```

---

## Uniformity test for circular data

*Uniformity tests for circular data.*

---

### Description

Hypothesis tests of uniformity for circular data.

### Usage

```
kuiper(u, rads = FALSE, R = 1)
watson(u, rads = FALSE, R = 1)
```

### Arguments

| | |
|---|---|
| `u` | A numeric vector containing the circular data, which cna be expressed in degrees or radians. |
| `rads` | A boolean variable. If the data are in radians, put this TRUE. If the data are expressed in degrees make this FALSE. |
| `R` | If R = 1 the asymptotic p-value will be calcualted. If R is greater than 1 the bootstrap p-value is returned. |

### Details

The high concentration (hcf.circaov), log-likelihood ratio (lr.circaov), embedding approach (embed.circaov) or the non equal concentration parameters approach (het.circaov) is used.

### Value

This is an "htest"class object. Thus it returns a list including:

| | |
|---|---|
| `statistic` | The test statistic value. |
| `parameter` | This is usually the degrees of freedom of the test, but here this is "NA" because the asymptotic based p-value is computed in a different way or because bootstrap was employed. |
| `p.value` | The p-value of the test. |
| `alternative` | A character with the alternative hypothesis. |
| `method` | A character with the test used. |
| `data.name` | A character vector with two elements. |

### Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <`mtsagris@uoc.gr`> and Giorgos Athineou <gioathineou@gmail.com>.

### References

Jammalamadaka, S. Rao and SenGupta, A. (2001). Topics in Circular Statistics, pg. 153–55 (Kuiper's test) and pg. 156–157 (Watson's test).

### See Also

`rayleigh`, `ptest`, `vmf.mle`, `rvonmises`

### Examples

```
x <- rvonmises(n = 40, m = 2, k = 10)
kuiper(x, rads = TRUE)
watson(x, rads = TRUE)
x <- rvonmises(40, m = 2, k = 0)
kuiper(x, rads = TRUE)
watson(x, rads = TRUE)
```

---

`von Mises kernel density estimation` *Kernel density estimation of circular data with a von Mises kernel*

---

## Description

Kernel density estimation of circular data with a von Mises kernel.

## Usage

```
vm.kde(u, h, thumb = "none", rads = TRUE)
```

## Arguments

| | |
|---|---|
| `u` | A numeric vector containing the data. |
| `h` | The bandwidth. |
| `thumb` | It can be either "none", so the bandwidth the user has set will be used, "tay" for the method of Taylor (2008) or "rot" for the method of Garcia-Portugues (2013). |
| `rads` | If the data are in radians, this should be TRUE and FALSE otherwise. |

## Details

The user has the option to use a bandwidth he/she has found in some way (cross-validation) or estimate it as Taylor (2008) or Garcia-Portugues (2013).

## Value

A list including:

| | |
|---|---|
| `h` | The bandwidth. If the user chose one of "tay" or "rot" the estimated bandwidth will be returned. |
| `f` | The kernel density estimate at the observed points. |

## Author(s)

Michail Tsagris.

R implementation and documentation: Michail Tsagris <mtsagris@uoc.gr> and Giorgos Athineou<gioathineou@gmail.com>.

## References

Taylor, C. C. (2008). Automatic bandwidth selection for circular density estimation. Computational Statistics & Data Analysis, 52(7): 3493-3500.

Garcia Portugues, E. (2013). Exact risk improvement of bandwidth selectors for kernel density estimation with directional data. Electronic Journal of Statistics, 7, 1655-1685.

### See Also

vmkde.tune, vmfkde.tune, vmf.kde

### Examples

```
x <- rvonmises(100, 2.4, 10, rads = TRUE)
hist(x, freq = FALSE)
f1 <- vm.kde(x, h = 0.1, thumb = "rot", rads = TRUE)$f
f2 <- vm.kde(x, h = 0.1, thumb = "tay", rads = TRUE)$f
h <- vmkde.tune(x)[1]
f3 <- vm.kde(x, h = h, thumb = "none", rads = TRUE)$f
points(x, f1, col = 1)
points(x, f2, col = 2)
points(x, f3, col = 3)
```

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## von Mises-Fisher kernel density estimation for (hyper-)spherical data

*Kernel density estimation for (hyper-)spherical data using a von Mises-Fisher kernel*

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### Description

A von Mises-Fisher kernel is used for the non parametric density estimation.

### Usage

```
vmf.kde(x, h, thumb = "none")
```

### Arguments

| | |
|---|---|
| x | A matrix with unit vectors, i.e. the data being expressed in Euclidean cordinates. |
| h | The bandwidth to be used. |
| thumb | If this is "none", the given bandwidth is used. If it is "rot" the rule of thumb suggested by Garcia-Portugues (2013) is used. |

### Details

A von Mises-Fisher kernel is used for the non parametric density estimation.

### Value

A list including:

| | |
|---|---|
| h | The bandwidth used. |
| f | A vector with the kernel density estimate calculated for each of the data points. |

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### References

Garcia Portugues, E. (2013). Exact risk improvement of bandwidth selectors for kernel density estimation with directional data. Electronic Journal of Statistics, 7, 1655–1685.

### See Also

vmfkde.tune, vm.kde, vmf.mle, vmkde.tune

### Examples

```
x <- rvmf(100, rnorm(5), 15)
h <- vmfkde.tune(x)[1]
f1 <- vmf.kde(x, h = h, thumb = "none")
f2 <- vmf.kde(x, h = h, thumb = "rot")
f1$h  ;  f2$h
```

# Index